Approximate Universal Artificial Intelligence

A Monte-Carlo AIXI Approximation

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General Reinforcement Learning Problem

Worst case scenario: Environment is unknown. Observations may be noisy. Effects of actions may be stochastic. No explicit notion of state. Perceptual aliasing. Rewards may be sparsely distributed.

Notation:

- Agent interacts with an unknown environment μ by making actions $a \in \mathcal{A}$.
- Environment responds with observations $o \in \mathcal{O}$ and rewards $r \in \mathcal{R}$. For convenience, we sometimes use $x \in \mathcal{O} \times \mathcal{R}$.
- $x_{1:n}$ denotes $x_1, x_2, \dots x_n$, $x_{<n}$ denotes $x_1, x_2, \dots x_{n-1}$ and $ax_{1:n}$ denotes $a_1, x_1, a_2, x_2, \dots a_n, x_n$.

MC-AIXI-CTW in context

Some approaches to (aspects of) the general reinforcement learning problem:

- Model-free RL with function approximation (e.g. TD)
- POMDP (assume an observation / transition model, maybe learn parameters?)
- Learn some (hopefully compact) state representation, then use MDP solution methods

Our approach:

- Directly approximate AIXI, a universal Bayesian optimality notion for general reinforcement learning agents.

AIXI: A Bayesian Optimality Notion

$$a_t^{AIXI} = \arg\max_{a_t} \sum_{x_t} \ldots \max_{a_{t+m}} \sum_{x_{t+m}} \left[\sum_{i=t}^{t+m} r_i \right] \sum_{\rho \in \mathcal{M}} 2^{-K(\rho)} \rho(x_{1:t+m} | a_{1:t+m}),$$

- Expectimax + (generalised form of) Solomonoff Induction
- Model class $\mathcal{M}$ contains all enumerable chronological semi-measures.
- Kolmogorov Complexity used as an Ockham prior.
- $m := b - t + 1$ is the "remaining search horizon". b is the maximum age of the agent

Caveat: Incomputable. Not an algorithm!

Describing Environments, AIXI Style

- A *history* h is an element of $(\mathcal{A} \times \mathcal{X})^* \cup (\mathcal{A} \times \mathcal{X})^* \times \mathcal{A}$.
- An *environment* ρ is a sequence of conditional probability functions $\{\rho_0, \rho_1, \rho_2, \ldots\}$, where for all $n \in \mathbb{N}$, $\rho_n \colon \mathcal{A}^n \to \mathit{Density}\ (\mathcal{X}^n)$ satisfies

$$\forall a_{1:n} \forall x_{<n} : \ \rho_{n-1}(x_{<n}|a_{<n}) = \sum_{x_n \in \mathcal{X}} \rho_n(x_{1:n}|a_{1:n}), \rho_0(\epsilon|\epsilon) = 1.$$

- The ρ-probability of observing x_n in cycle n given history $h = ax_{<n}a_n$ is

$$\rho(x_n|ax_{<n}a_n) := \frac{\rho(x_{1:n}|a_{1:n})}{\rho(x_{<n}|a_{<n})}$$

provided $\rho(x_{<n}|a_{<n}) > 0$.

Learning a Model of the Environment

We will be interested in agents that use a *mixture environment model* to learn the true environment μ.

$$\xi(x_{1:n}|a_{1:n}) := \sum_{\rho \in \mathcal{M}} w_0^\rho \rho(x_{1:n}|a_{1:n})$$

- $\mathcal{M} := \{\rho_1, \rho_2, \ldots\}$ is the model class
- w_0^ρ is the prior weight for environment ρ.
- Satisfies the definition of an environment model. Therefore, can predict by using:

$$\xi(x_n|ax_{<n}a_n) = \sum_{\rho \in \mathcal{M}} w_{n-1}^\rho \rho(x_n|ax_{<n}a_n), \;\; w_{n-1}^\rho := \frac{w_0^\rho \rho(x_{<n}|a_{<n})}{\sum_{\nu \in \mathcal{M}} w_0^\nu \nu(x_{<n}|a_{<n})}$$

Theoretical Properties

Theorem: Let μ be the true environment. The μ-expected squared difference of μ and ξ is bounded as follows. For all $n \in \mathbb{N}$, for all $a_{1:n}$,

$$\sum_{k=1}^{n} \sum_{x_{1:k}} \mu(x_{<k}|a_{<k}) \Big(\mu(x_k|ax_{<k}a_k) - \xi(x_k|ax_{<k}a_k) \Big)^2 \leq$$

$$\min_{\rho \in \mathcal{M}} \left\{ -\ln w_0^\rho + D_{KL}(\mu(\cdot|a_{1:n}) \,\|\, \rho(\cdot|a_{1:n})) \right\},$$

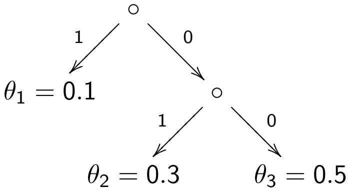
where $D_{KL}(\cdot \,\|\, \cdot)$ is the KL divergence of two distributions.

Roughly: The predictions made by ξ will converge to those of μ if a model close (w.r.t. KL Divergence) to μ is in $\mathcal{M}$.

Prediction Suffix Trees

A prediction suffix tree is a simple, tree based variable length Markov model. For example, using the PST below, having initially been given data 01:

$$\begin{aligned}
\Pr(010|01) &= \Pr(0|01) \times \Pr(1|010) \times \Pr(0|0101) \\
&= (1-\theta_1)\theta_2(1-\theta_1) \\
&= 0.9 * 0.3 * 0.9 \\
&= 0.243
\end{aligned}$$



Context Tree Weighting

- Context Tree Weighting is an online prediction method.
- CTW uses mixture of prediction suffix trees.
- Smaller suffix trees are given higher initial weight, which helps to avoid overfitting when data is limited.
- Let $\mathcal{C}_D$ denote the class of all prediction suffix trees of maximum depth D, then CTW computes in time $O(D)$:

$$\Pr(x_{1:t}) = \sum_{M \in \mathcal{C}_D} 2^{-\Gamma_D(M)} \Pr(x_{1:t}|M)$$

- $\Gamma_D(M)$ is description length of context tree M.
- This is truly amazing, as computing the sum naively would take time double-exponential in D!

Model Class Approximation

Action-Conditional Context Tree Weighting Algorithm:

Approximate model class of AIXI with a mixture over *all* action-conditional Prediction Suffix Tree structures of maximum depth D.

- PSTs are a form of variable order Markov model.
- Context Tree Weighting algorithm can be adapted to compute a mixture of over $2^{2^{D-1}}$ environment models in time $O(D)$!
- Inductive bias: smaller PST structures favoured.
- PST parameters are learnt using KT estimators.
  KL-divergence term in previous theorem grows $O(\log n)$.
- Intuitively, efficiency of CTW is due to clever exploitation of shared structure.

Greedy Action Selection

- Action $a \in \mathcal{A}$ has value $V(a) = \mathbf{E}[R|a] =$ expected return.
- Consider Bandit setting: No history or state dependence.
- Optimal action/arm: $a^* := \arg\max_a V(a)$ (unknown).
- $V(a)$ unknown $\Rightarrow$ frequency estimate
  $\hat{V}(a) := \frac{1}{T(a)} \sum_{t:a_t=a} R_t, \quad R_t =$ actual return at time t.
  $T(a) := \#\{t \leq T : a_t = a\} = \#$times arm a taken so far.
- Greedy action: $a_{T+1}^{greedy} = \arg\max_a \hat{V}(a)$
- Problem: If a^* accidentally looks bad (low early $\hat{V}(a)$),
  it will never be taken again = explore/exploit dilemma.
- Solution: Optimism in the face of uncertainty ...

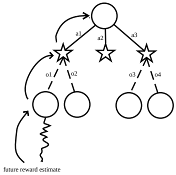
Upper Confidence Algorithm for Bandits

- UCB action: $a^{UCB}_{T+1} := \arg\max_a V^+(a)$
  $V^+(a) := \hat{V}(a) + C\sqrt{\frac{\log T}{T(a)}}$, $\quad C > 0$ suitable constant.
- If arm under-explored (i.e. $T(a) \ll \log T$)
  $\Rightarrow$ $V^+(a)$ huge $\Rightarrow$ UCB will take arm a
  $\Rightarrow$ Every arm taken infinitely often $\Rightarrow$ $\hat{V}(a) \to V(a)$
- If sub-optimal arm over-explored (i.e. $T(a) \gg \log T$)
  $\Rightarrow V^+(a) \approx \hat{V}(a) \to V(a) < V(a^*) \leftarrow \hat{V}(a^*) < V^+(a^*)$
  $\Rightarrow$ UCB will **not** take arm a
- Fazit: $T(a) \propto \log T$ for all suboptimal arms.
  $T(a^*) = T - O(\log T)$, i.e. only $O(\log T) \ll T$ subopt. actions
- UCB is theor. optimal explore/exploit strategy for Bandits.

  Application: Use "heuristically" in expectimax tree search...

Expectimax Approximation

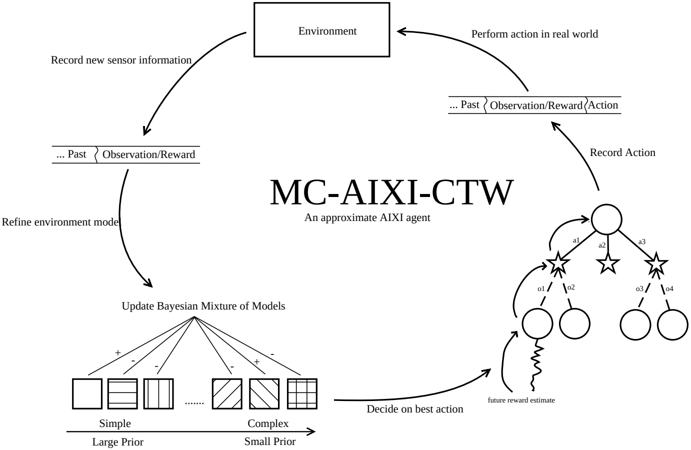
Monte Carlo approximation of expectimax Tree Search (MCTS)
Upper Confidence Tree (UCT) algorithm:



- Sample observations from CTW distribution.
- Select actions with highest
  Upper Confidence Bound (UCB) V^+.
- Expand tree by one leaf node (per trajectory).
- Simulate from leaf node further down using (fixed) playout policy.
- Propagate back the value estimates for each node.
  Repeat until timeout.

- With sufficient time, converges to the expectimax solution.
- Value of Information correctly incorporated
  when instantiated with a mixture environment model.
- Gives Bayesian solution to the exploration/exploitation dilemma.

Agent Architecture (MC-AIXI-CTW = UCT+CTW)



Relationship to AIXI

Given enough thinking time, MC-AIXI-CTW will choose:

$$a_t = \arg\max_{a_t} \sum_{x_t} \cdots \max_{a_{t+m}} \sum_{x_{t+m}} \left[\sum_{i=t}^{t+m} r_i \right] \sum_{M \in \mathcal{C}_D} 2^{-\Gamma_D(M)} \Pr(x_{1:t+m} | M, a_{1:t+m})$$

In contrast, AIXI chooses:

$$a_t = \arg\max_{a_t} \sum_{x_t} \ldots \max_{a_{t+m}} \sum_{x_{t+m}} \left[\sum_{i=t}^{t+m} r_i \right] \sum_{\rho \in \mathcal{M}} 2^{-K(\rho)} \Pr(x_{1:t+m} | a_{1:t+m}, \rho)$$

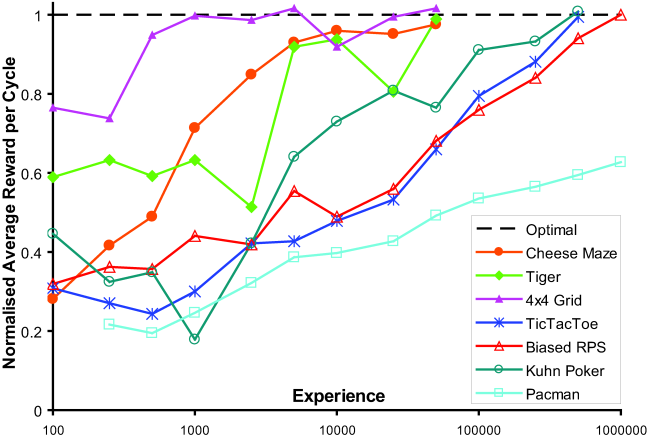
Algorithmic Considerations

- Restricted the model class to gain the desirable computational properties of CTW
- Approximated the finite horizon expectimax operation with a MCTS procedure
- $O(Dm \log(|\mathcal{O}||\mathcal{R}|))$ operations needed to generate m observation/reward pairs (for a single simulation)
- $O(tD \log(|\mathcal{O}||\mathcal{R}|))$ space overhead for storing the context tree.
- Anytime search algorithm
- Search can be parallelized
- $O(D \log(|\mathcal{O}||\mathcal{R}|)))$ to update the context tree online

Experimental Setup

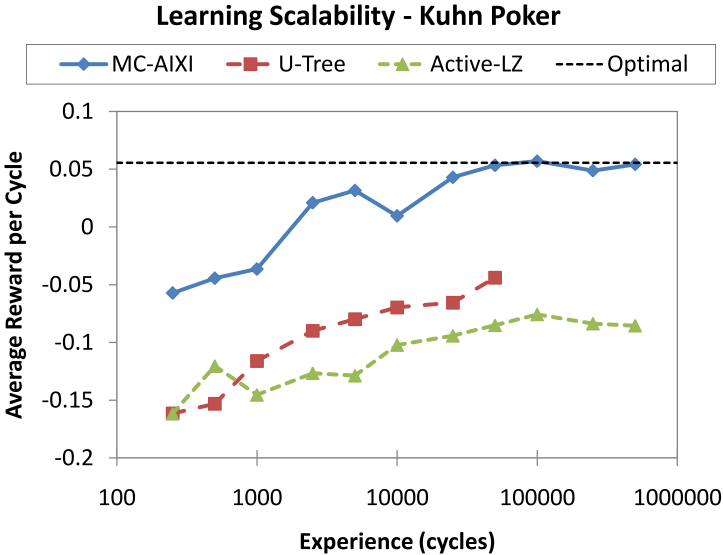
- Agent tested on a number of POMDP domains, as well as TicTacToe and Kuhn Poker.
- Agent required to *both* learn *and* plan.
- The context depth and search horizon were made as large as possible subject to computational constraints.
- ϵ-Greedy training, with a decaying ϵ
- Greedy evaluation

Results



Normalised Average Reward per Cycle
0
0.2
0.4
0.6
0.8
1
100
1000
10000
100000
1000000
Experience
Optimal
Cheese Maze
Tiger
4x4 Grid
TicTacToe
Biased RPS
Kuhn Poker
Pacman

Comparison to Other RL Algorithms



Resources Required for (Near)Optimal Performance

Domain	Experience	Simulations	Search Time
Cheese Maze	5×10^4	500	0.9s
Tiger	5×10^4	10000	10.8s
4×4 Grid	2.5×10^4	1000	0.7s
TicTacToe	5×10^5	5000	8.4s
Biased RPS	1×10^6	10000	4.8s
Kuhn Poker	5×10^6	3000	1.5s

- Timing statistics collected on an Intel dual quad-core 2.53Ghz Xeon.
- Toy problems solvable in reasonable time on a modern workstation.
- General ability of agent will scale with better hardware.

Limitations

- PSTs inadequate to represent many simple models compactly. For example, it would be unrealistic to think that the current MC-AIXI-CTW approximation could cope with real-world image or audio data.
- Exploration/exploitation needs more attention. Can something principled *and* efficient be done for general Bayesian agents using large model classes?

Future Work

- Uniform random rollout policy used in ρUCT.
  A learnt policy should perform much better.
- All prediction was done at the bit level. Fine for a first attempt, but no need to work at such a low level.
- Mixture environment model definition can be extended to continuous model classes.
- Incorporate more (action-conditional) Bayesian machinery.
- Richer notions of context.

References

- For more information, see:

  A Monte-Carlo AIXI Approximation (2011),
  J. Veness, K.S. Ng, M. Hutter, W. Uther, D. Silver
  http://dx.doi.org/10.1613/jair.3125

  Highlights: a direct comparison to U-Tree / Active-LZ, improved model class approximation (FAC-CTW) and more relaxed presentation.

- Video of the latest version playing Pacman
  http://www.youtube.com/watch?v=yfsMHtmGDKE