Indefinitely Oscillating Martingales

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Abstract

We construct a class of nonnegative martingale processes that oscillate indefinitely with high probability. For these processes, we state a uniform rate of the number of oscillations and show that this rate is asymptotically close to the theoretical upper bound. These bounds on probability and expectation of the number of upcrossings are compared to classical bounds from the martingale literature. We discuss two applications. First, our results imply that the limit of the minimum description length operator may not exist. Second, we give bounds on how often one can change one's belief in a given hypothesis when observing a stream of data.*

* Jan Leike and Marcus Hutter. *Indefinitely Oscillating Martingales.* Extended Technical Report. `http://arxiv.org/abs/1408.3169`. The Australian National University, 2014.

Overview

- P, Q probability measures on Σ^ω
  $\iff Q/P$ is nonnegative P-martingale
- For $Q := P(\,\cdot \mid H)$: oscillations = mind changes
  $\implies$ Bounds on the probability of mind changes:

$$P[\text{ change mind } 2k \text{ times by at least } \alpha\ ] \leq \left(\frac{1-\alpha}{1+\alpha}\right)^k$$

- There are martingales that oscillate indefinitely
  $\implies \lim_{|u|\to\infty} \mathrm{MDL}^u$ may not exist,
  i.e., MDL may be inductively inconsistent
- Upper bounds for the rate of oscillations

Outline

Measures and Martingales

Definition (Martingale)

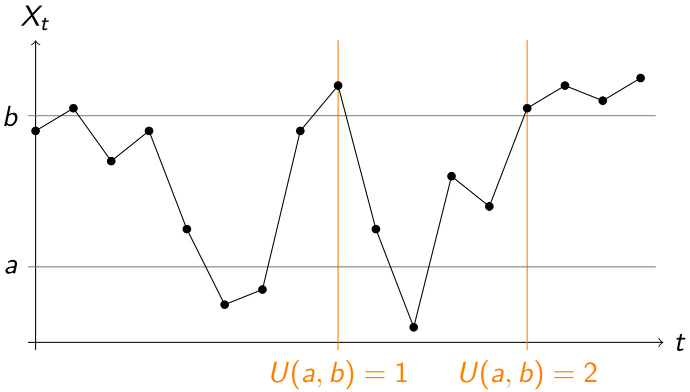
$(X_t)_{t \in \mathbb{N}}$ is *P-martingale* iff $\mathbb{E}_P[X_{t+1} \mid X_t] = X_t$.

Theorem (Measures and Martingales)

- *P is a probability measure on infinite strings*
- *$X_t(v) = Q(v_{1:t})/P(v_{1:t})$ for P-almost all $v \in \Sigma^\omega$*

Q is a probability measure on infinite strings, *Q is absolutely continuous with respect to P on cylinder sets*	$\iff$	*$(X_t)_{t \in \mathbb{N}}$ is nonnegative P-martingale,* $\mathbb{E}[X_t] = 1$

Upcrossings



$U(a,b) :=$ number of times X_t falls below a and subsequently rises above b

Doob's and Dubins' Bounds

- $(X_t)_{t\in\mathbb{N}}$ is nonnegative P-martingale
- $\mathbb{E}[X_0] = 1$

Theorem (Dubins' Inequality)

$$P\big[U(a,b) \geq k\big] \leq \frac{a^k}{b^k}$$

Theorem (Doob's Upcrossing Inequality)

$$\mathbb{E}\big[U(a,b)\big] \leq \frac{a}{b-a}$$

Bounds on Mind Changes

What is the probability of changing you mind k times by more than $\alpha > 0$?

- $H \subseteq \Sigma^\omega$ is some hypothesis
- $P(H \mid v_{1:t})$ is belief in H after observing $v_{1:t}$

$$P(H \mid v_{1:t}) \propto \frac{P(v_{1:t} \mid H)}{P(v_{1:t})}$$

$Q := P(\,\cdot \mid H)$ is measure $\Longrightarrow P(H \mid v_{1:t})$ is P-martingale.

Results from Martingales

$A(\alpha) :=$ number of changes of $P(H \mid v_{1:t})$ by at least α.

Theorem (Davis 2013†)

$$P\big[A(\alpha) \geq 2k\big] \leq \left(\frac{1-\alpha}{1+\alpha}\right)^{2k}$$

Using martingale theory:

Theorem

$$P\big[A(\alpha) \geq 2k\big] \leq \left(\frac{1-\alpha}{1+\alpha}\right)^{k} \qquad \mathbb{E}\big[A(\alpha)\big] \leq \frac{1}{\alpha}$$

† Ernest Davis. *Bounding changes in probability over time: It is unlikely that you will change your mind very much very often.* Technical Report. `https://cs.nyu.edu/davise/papers/dither.pdf`. 2013.

MDL Convergence

Definition (Minimal Description Length)

$$\mathrm{MDL}^{v_{1:t}} := \operatorname*{arg\,min}_{Q \in \mathcal{M}} \left\{ -\log Q(v_{1:t}) + K(Q) \right\}$$

Theorem (MDL May Be Inductively Inconsistent)

For any P with perpetual entropy there is a class $\mathcal{M} \ni P$ such that

$$\lim_{t \to \infty} \mathrm{MDL}^{v_{1:t}} \text{ does not exist}$$

with P-probability $1 - \varepsilon$.

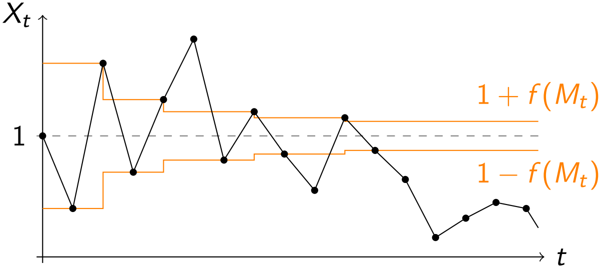
An Indefinitely Oscillating Martingale

Theorem (Lower Bound)

- $f : \mathbb{N} \to [0,1)$ *monotone decreasing,* $\sum_{i=1}^{\infty} f(i) < \delta/2$
- *P has perpetual entropy*

$\Longrightarrow \exists$ *nonnegative P-martingale with*

$$P\left[\forall m.\ U(1 - f(m), 1 + f(m)) \geq m\right] \geq 1 - \delta.$$



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Theorem (Upper Bound)

- $f : \mathbb{N} \to [0,1)$ *monotone decreasing,* $\sum_{i=1}^{\infty} f(i) = \infty$

$\Longrightarrow \forall$ *nonnegative P-martingale*

$$P\left[\forall m.\ U(1 - f(m), 1 + f(m)) \geq m\right] = 0.$$

Additional Material

Concrete Bounds

Corollary (Concrete Lower Bound)

- *P has perpetual entropy*

$\Longrightarrow \exists$ *nonnegative P-martingale with*

$$P\left[\forall\varepsilon > 0.\ U(1-\varepsilon, 1+\varepsilon) \in \Omega\left(\frac{\delta}{\varepsilon(\ln\varepsilon)^2}\right)\right] \geq 1-\delta$$

$$\mathbb{E}\left[U(1-\varepsilon, 1+\varepsilon)\right] \in \Omega\left(\frac{1}{\varepsilon(\ln\varepsilon)^2}\right)$$

$$P\left[\forall\varepsilon < 0.015.\ U(1-\varepsilon, 1+\varepsilon) > \frac{\delta}{\varepsilon(\ln\varepsilon)^2}\right] \geq 1-\delta$$

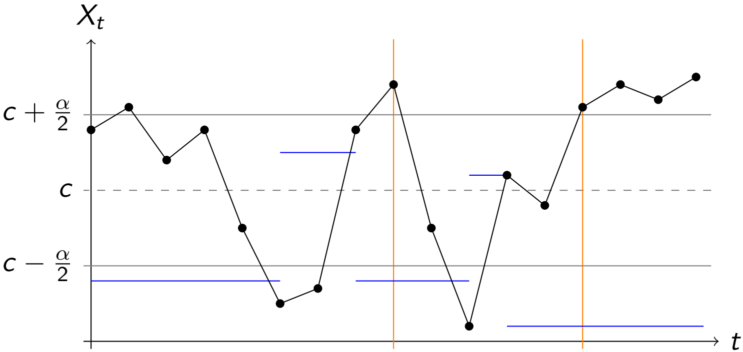
$$\forall\varepsilon < 0.015 \quad \mathbb{E}\left[U(1-\varepsilon, 1+\varepsilon)\right] > \frac{\delta(1-\delta)}{\varepsilon(\ln\varepsilon)^2}$$

Corollary (Concrete Upper Bound)

$\forall$ *nonnegative P-martingale*

$$\forall a, b > 0 \quad P\left[\forall\varepsilon > 0.\ U(1-\varepsilon, 1+\varepsilon) \geq \frac{a}{\varepsilon\log(1/\varepsilon)} - b\right] = 0.$$

Alternations vs. Upcrossings



- Alternations: drift over time
- Alternations: the initial value matters
- One upcrossing = two alternations