Context Tree Weighting

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Motivation

- Context Tree Weighting (CTW) is a Bayesian Mixture of the huge class of variable-order Markov processes.
- It is principled and computationally efficient.
- It leads to excellent predictions and compression in practice and in theory.
- It can (and will) be used to approximate Solomonoff's universal prior $\xi_U(x)$.

Prediction of I.I.D Sequences

- Suppose that we have a sequence which we believe to be I.I.D. but we do not know the probabilities.

- If x has been observed n_x times, then we can use the (generalized Laplace rule, Dirichlet(α) prior) estimate $Pr(x_{n+1} = x | x_{1:n}) = \frac{n_x + \alpha}{n + M\alpha}$, where M is the size of the alphabet and $\alpha > 0$ is a smoothing constant.

- We use the special case of binary alphabet and $\alpha = 1/2$ (Jeffrey's=Beta(1/2) prior, $\approx$ minimax optimal).

- The probability of a 0 if we have previously seen a zeros and b ones hence is $\frac{a+1/2}{a+b+1}$ and the probability of a 1 is $\frac{b+1/2}{a+b+1}$

Joint Prob. for I.I.D with Beta(1/2) Prior

- The joint prob. of sequence is product of individual probabilities independent of order: $Pr(x_1, .., x_n) = Pr(x_{\pi(1)}, ..., x_{\pi(n)})$ $\forall$ permutations π.
- We denote the probability of a zeros and b ones with $P_{kt}(a,b)$
- $P_{kt}(a+1,b) = P_{kt}(a,b)\frac{a+1/2}{a+b+1}, \qquad P_{kt}(0,0) = 1.$
- $P_{kt}(a,b+1) = P_{kt}(a,b)\frac{b+1/2}{a+b+1}, \qquad Pr(x_{1:n}) = P_{kt}(a,b).$

$a\backslash b$	0	1	2	3	4	...
0	1	1/2	3/8	5/16	35/128	...
1	1/2	1/8	1/16	5/128	7/256	...
2	3/8	1/16	3/128	3/256	7/1024	...
3	5/16	5/128	3/256	5/1024	5/2048	...

- Example: $Pr_{kt}(0011) = \frac{1/2}{1} \cdot \frac{1+1/2}{2} \cdot \frac{1/2}{3} \cdot \frac{1+1/2}{4} = Pr_{kt}(0101) = Pr_{kt}(1001) = ... = Pr_{kt}(2,2) = \frac{3}{128}$
- Direct: $Pr(x_{1:n}) = \prod_{t=1}^{n} Pr(x_t|x_{<t}) = P_{kt}(a,b) = \int_\theta Pr_\theta(x)\text{Beta}_{1/2}(\theta)d\theta = \frac{1}{\pi}\int \theta^{a-1/2}(1-\theta)^{b-1/2}d\theta = \frac{(a-1/2)!(b-1/2)!}{(a+b)!\pi}$

Using Context

- If you hear a word that you believe is either "cup" or "cop", then you can use context to decide which one it was.
- If you are sure they said "Wash the" just before, then it is probably "cup".
- If they said "Run from the", it might be "cop".
- We will today look at models, which have a short term memory dependence.
- In other words models that remember the last few observations.

Markov Models

- Suppose that we have a sequence x_t of elements from some finite alphabet
- Suppose that $Pr(x_t \mid x_{1:t-1}) = Pr(x_t|x_{t-1})$ is always true, then we have a Markov Model. At most the latest observation matters
- If $Pr(x_t \mid x_{1:t-1}) = Pr(x_t|x_{t-k:t-1})$, then we have a k:th order Markov Model. At most the k latest observations matter
- Given that we have chosen k and a (several) sequence(s) of observations, then for every context (string of length k) we can estimate probabilities for all possible next observations using the KT-estimator.
- We define the probability of the event of having a 0 after s given that we have, in this context, previously seen a_s zeros and b_s ones to be $\frac{a_s+1/2}{a_s+b_s+1}$ and the probability of a one to be $\frac{b_s+1/2}{a_s+b_s+1}$
- Formally: $[x]_{|s} := (x_t : x_{t-k:t-1} = s) \equiv$ all those x_t with k-context $x_{t-k:t-1} = s$. $a_s = \#\{0 \text{in} [x]_{|s}\}$, $b_s = \#\{1 \text{in} [x]_{|s}\}$.

Markov-KT Examples

- Compute $Pr_k(x) = (\frac{1}{2})^k \prod_{s\in\{0,1\}^k} P_{kt}(a_s, b_s)$ for Example $x = 0010100111101010$.

- $k = 0:$ $a = b = 8$.
  $Pr_0(0010100111101010) = P_{kt}(8,8) \approx 402 \times 2^{-27}$

- $k = 1:$ $a_0 = 2,\ b_0 = 5,\ a_1 = 5,\ b_1 = 3$.
  $Pr_1(0\ 010100111101010) = \frac{1}{2} P_{kt}(2,5) P_{kt}(5,3) =$
  $\frac{1}{2} \cdot \frac{9}{2048} \cdot \frac{45}{32768} = 405 \cdot 2^{-27}$

- $k = 2:$

s	00	01	10	11
a_s	0	4	1	1
b_s	2	1	3	2
$[x]_{\mid s}$	11	00100	1011	110

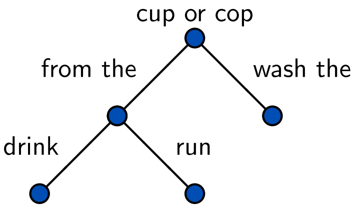
  $Pr_2(00\ 10100111101010) =$
  $(\frac{1}{2})^2 P_{kt}(0,2) P_{kt}(4,1) P_{kt}(1,3) P_{kt}(1,2)$
  $= \frac{1}{4} \cdot \frac{3}{8} \cdot \frac{7}{256} \cdot \frac{5}{16} \cdot \frac{1}{16} = 105 \cdot 64 \cdot 2^{-27}$

Choosing k

- If we decide to model our sequence with a k:th order Markov model we then need to pick k.
- How long dependencies exist?
- Which contexts have we seen enough to make good predictions from? Shorter contexts appear more often.
- Solution 1: Use MDL to select k.
- Solution 2: We can take the Bayesian approach and have a mixture over all orders.
- We can choose a prior (initial mixture weights) that favors shorter orders (simpler models), e.g. $P(k) = \frac{1}{k(k+1)}$.

Context Trees

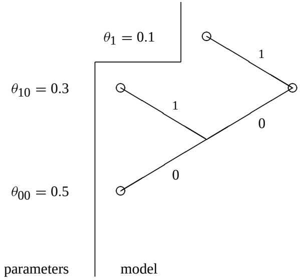
- It can be natural to consider contexts of different lengths in a model, depending on the situation.
- For example if we have heard "from the" and want to decide if the next word is "cup" or "cop" it is useful to also know if it is "drink from the" or "run from the" while if we have the context "wash the" it might be enough.
- Having redundant parameters lead to a need for more data to find good parameter estimates. With small amount of data for a context, it is better to be shallow.



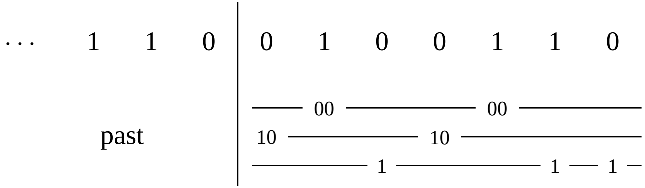
Tree Source

Example: Tree $\mathcal{T} \mathrel{\widehat{=}} \{00, 10, 0\}$

$$P(x_t = 1|...x_{t-1} = 1) = \theta_1$$
$$P(x_t = 1|...x_{t-2} = 0, x_{t-1} = 0) = \theta_{00}$$
$$P(x_t = 1|...x_{t-2} = 1, x_{t-1} = 0) = \theta_{10}$$
.



Contexts $\mathcal{T}$ in sequence 01001100 with past ...110:



$P_{\mathcal{T}}(0100110|...110) = \frac{3}{8} \cdot \frac{3}{8} \cdot \frac{1}{16}$, where $\frac{3}{8}$, $\frac{3}{8}$, $\frac{1}{16}$ are the P_{kt} probabilities of subsequences 11,00,010 corresponding to leaves/contexts 00,10,1, respectively.

Context Tree Weighting (CTW)

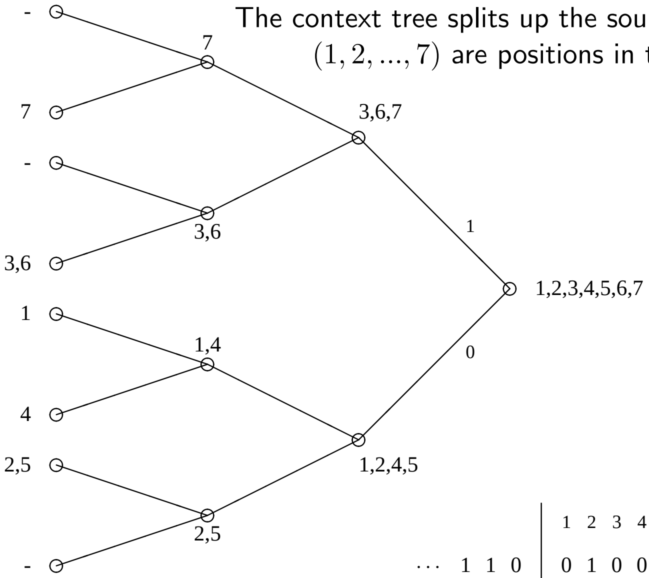
- Given a tree we have defined contexts such that we are always exactly in one context. Given data we estimate probabilities for the next observation given the context.
- We do not know the most appropriate tree. So we have to estimate it or Bayes-average over all trees.
- The number of trees (and weights) increases double exponentially with the depth.
- The CTW algorithm resolves this problem.
- It stores two numbers (for the binary case) per node in the full tree of the given maximal depth and calculates a third through a recursive formula.
- The stored numbers are just counts and are easy to update.

Context Tree Weighting

- To define a probability of a finite sequence, CTW uses a recursive formula that starts at the leafs of the tree and moves towards the root where the probability of the whole sequence will appear.
- The next few slides will explain it with an example.
- Every node in the full tree corresponds to a context s, e.g. 010.
- We let a_s be the number of times that a 0 has followed s and b_s is the number of times 1 has followed.
- For every context that corresponds to a node in the full tree we will define a number P_s. $P_{root} \equiv P_\epsilon \equiv P_{CTW}$ will be the probability for the whole sequence.
- For a leaf we let $P_s = P_{kt}(a_s, b_s)$.
- Not leaf: $P_s = \frac{1}{2}(P_{kt}(a_s, b_s) + P_{0s}P_{1s})$

Context Tree Splits – Example

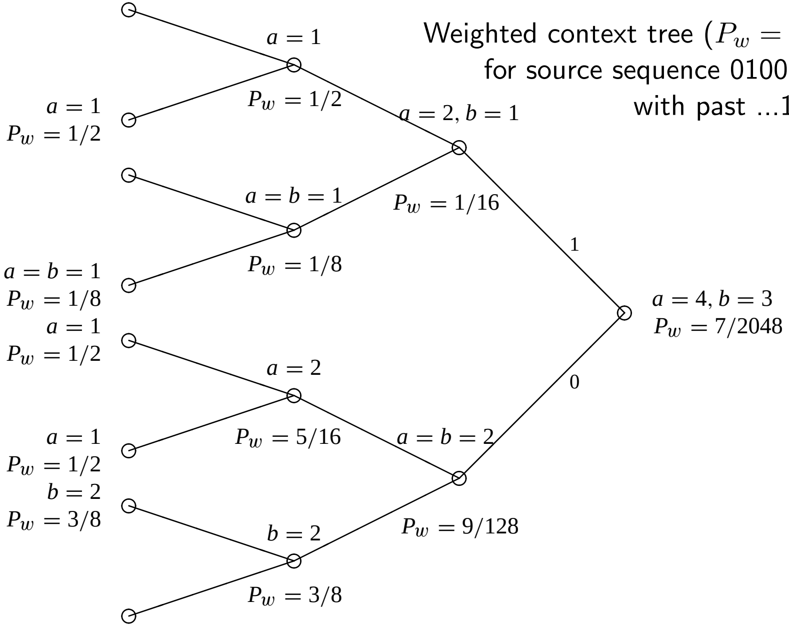
The context tree splits up the source sequence.
$(1, 2, ..., 7)$ are positions in the sequence.



			1	2	3	4	5	6	7
···	1	1	0	0	1	0	0	1	1	0
		past	source sequence						

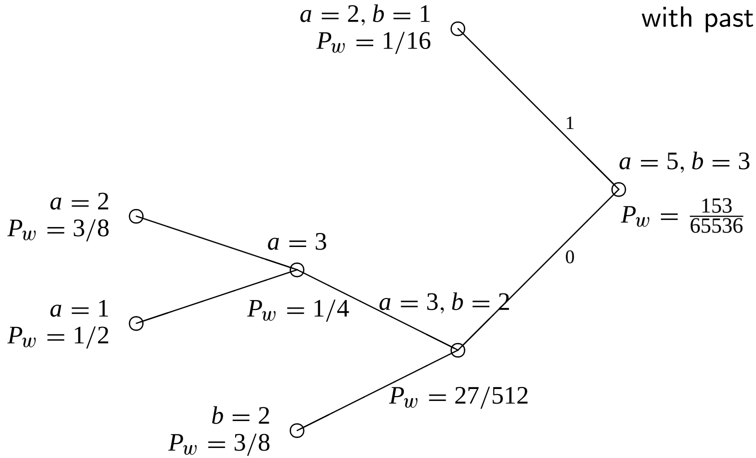
Weighted Context Tree – Example

Weighted context tree ($P_w = P_s$)
for source sequence 0100110
with past ...110.



Weighted Context Tree Update–Example

Updated path of weighted context tree
for 0100110 followed by 0
with past ...110.

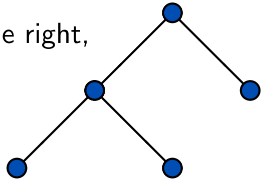


Coding Context Trees

- We can rewrite the probability P_{CTW} as a Bayesian mixture over trees with a prior based on a coding scheme for trees:
- Let $\mathcal{T}$ denote a tree and $CL(\mathcal{T})$ the code length for T
- Let tree prior $\Gamma(\mathcal{T}) = 2^{-CL(\mathcal{T})}$. Then

$$P_{CTW} = \sum_{\mathcal{T}} 2^{-CL(\mathcal{T})} \prod_{s \in \mathcal{T}} P_{kt}(a_s, b_s)$$

- The coding scheme is based on the following recursion:
  1. Code(tree) = Code(root-node)
  2. Code(Internal-node) = 1 Code(leaf-child) Code(right-child)
  3. Code(leaf-node) = 0
- 11000 is the code of the tree on the right, so the code length is 5.



Coding Redundancy

- Arithmetic Coding w.r.t. P_{CTW} gives code for x of length $CL_{CTW}(x) = \log 1/P_{CTW}(x)$.
- A coding scheme's redundancy is $CL(x_{1:T}) - \log 1/Pr(x_{1:T})$ where Pr is the true source.
- We are interested in the expected redundancy
- The expected redundancy, unlike the actual redundancy, is at least 0. The truth has the lowest (i.e. 0) expected redundancy.
- The Rissanen lower bound says that the expected redundancy is at least $\frac{1}{2}\log T$ per parameter.
- If the sequence is generated from a (stationary) tree (Markov) source $\mathcal{T}$ with S leafs, then for large enough T the expected redundancy of CTW is less than $CL(\mathcal{T}) + \frac{S}{2}\log T$.
- CTW achieves the Rissanen lower. bound
- This is essentially a corollary of the general (continuous+discrete) "entropy bounds" from UAI book (Hutter) / previous lectures.

Derivation of Coding Redundancy

The bound can be derived directly or follows from the generic bounds for Bayesian Sequence Prediction [Hut05]:

- Inst. Rel. Entropy: $d_t(x_{<t}) = \sum_{x_t \in \mathcal{X}} \mu(x_t|x_{<t}) \log \frac{\mu(x_t|x_{<t})}{\xi(x_t|x_{<t})}$
- Total Relative Entropy: $D_T = \sum_{t=1}^{T} \mathbb{E}[d_t]$ is the expected coding redundancy from using ξ instead of (the true) μ, i.e. $D_T = \mathbb{E}_\mu[CL_\xi(x_{1:T}) - CL_\mu(x_{1:T})]$
- Countable family of models: $D_T \leq \log w_\mu^{-1}$
- Continuous family of models:
- $D_T \leq \log w_\mu^{-1} + \frac{S}{2} \log \frac{T}{2\pi} + \frac{1}{2} \log \det j_T + O(1)$
- CTW is a continuous mixture with prior $w_\mu = 2^{-CL(\mathcal{T})} g(\theta)$ where $\mathcal{T}$ is the tree, θ represents the probability parameters and $g(\theta)$ is the Beta$(1/2, 1/2)$ prior density on the parameters.
- $\frac{1}{2} \log \det j_T = O(1)$ for tree sources so $D_T \leq CL(\mathcal{T}) + \frac{S}{2} \log T + O(1)$

Conclusion / Properties

- CTW is a simple algorithm for mixing contexts of different length to make predictions.
- The computational complexity for calculating the probability of a sequence is linear in the sequence length.
- CTW has good practical performance for many purposes, e.g. text compression.
- Given that the truth is a tree source there are bounds for how much worse CTW is than using the (unknown) true model.
- When the true model is not known there are limits (a lower bound) to how close we can be to the performance of the true model.
- CTW asymptotically achieves this theoretically optimal performance.

Literature

WST95 Willems, Shtarkov, and Tjalkens (1995) The Context-Tree Weighting Method: Basic Properties, 41. IEEE Transactions on Information Theory.

WST97 Willems, F.M.J., Shtarkov, Y.M. and Tjalkens, T.J. (1997). Reflections on the prize paper 'The context-tree weighting method: Basic properties'. Newsletters of the IEEE.

VNH+11 Hutter:11aixictwx J. Veness, K. S. Ng, M. Hutter, W. Uther, and D. Silver. A Monte Carlo AIXI approximation, Journal of Artificial Intelligence Research, 40:95–142, 2011.

Hut05 M. Hutter. Universal Artificial Intelligence: Sequential Decisions based on Algorithmic Probability. Springer, Berlin, 2005.