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Gluon Mass from Instantons

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Abstract

The gluon propagator is calculated in the instanton background in a form appropriate for extracting the momentum dependent gluon mass. In background-ξ-gauge we get for the mass 400MeV for small p^2 independent of the gauge parameter ξ.

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1. Introduction

In the following we will calculate the gluon propagator for different gauges. In background $\xi = 1$ gauge it has the simple form $S^{ab}_{\mu\nu} = \delta^{ab} g_{\mu\nu}/(p^2 - M(p)^2)$. In section 2 we will extract the inverse propagator from the terms quadratic in the fluctuations around a background field. In section 3 we will average this expression over relevant background fields and expand it for large momentum in a way to get the gluon mass $M(p)$. In section 4 we explicitly calculate $M_1 = M(p = \infty) = 420\text{MeV}$ in the instanton background. We will also see that there are further terms which do not vanish at large momentum but are more difficult to calculate. To get reliable results for large as well as for small momentum we make in section 5 an cluster expansion in the instanton density. For this purpose we need the gluon propagator in the 1 instanton background. The relevant formulas to construct this propagator are listed in section 6. Because it is a bit lengthy we will restrict ourself in section 7 to the case of small momentum. Till now as far as I know the instanton background is only treated at the classical level except for a calculation of the one loop action in the instanton background. This work should be seen as a first step to calculate gluonic quantum fluctuations around the instanton background and the simplest object to play with is the propagator. For high energies there are tools like operator product expansion with instantons included indirectly in the condensates [6]. This is another reason for the restriction to small momentum.

2. Gluon Propagator

The first task to obtain a formal expression for the gluon propagator in a background field is to expand $\mathcal{L}_{QCD}[\bar{A} + B]$ in the fluctuations B^a_μ around our background $\bar{A}^a_\mu$. The term quadratic in B^a_μ is then by definition the inverse gluon propagator. For $\bar{A}^a_\mu$ we will later use our instanton gas. We will use the QCD-Lagrangian

$$\mathcal{L}_{QCD} = \frac{1}{4g^2} G^a_{\mu\nu} G^{\mu\nu}_a \quad , \quad G^a_{\mu\nu}(A) = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + f_{abc} A^b_\mu A^c_\nu$$

where we have rescaled the fields in such a way that the coupling-constant-dependence is in front of $\mathcal{L}$. We will entirely work in Euclidian space with metric $\delta_{\mu\nu}$ instead of $g_{\mu\nu}$ because instantons do not make sense in Minkowski space. At the very end we can simply rotate back to Minkowski space. With these conventions

$$g^2 \mathcal{L}_{QCD}(\bar{A} + B) = \frac{1}{4} G^a_{\mu\nu}(\bar{A} + B) G^{\mu\nu}_a(\bar{A} + B) =$$

$$= \frac{1}{4} \overbrace{\bar{G}^a_{\mu\nu} \bar{G}^{\mu\nu}_a}^{O(B^0)} + \overbrace{B^a_\mu \bar{D}^{ab}_\nu \bar{G}^{\mu\nu}_a}^{O(B^1)} + \frac{1}{2} \overbrace{B^a_\mu (-\bar{D}^{ac}_\rho \bar{D}^{cb}_\rho \delta_{\mu\nu} - 2 f_{acb} \bar{G}^c_{\mu\nu} + \bar{D}^{ac}_\mu \bar{D}^{cb}_\nu) B^b_\nu}^{O(B^2)} +$$
$$+ \underbrace{f_{abc} B^b_\mu B^c_\nu \bar{D}^{ad}_\mu B^d_\nu}_{O(B^3)} + \frac{1}{4} \underbrace{f_{abc} B^b_\mu B^c_\nu f_{ade} B^d_\mu B^e_\nu}_{O(B^4)} + \partial_\mu(\ldots) \quad , \qquad (1)$$

where $\bar{D}^{ab}_\mu = \partial_\mu \delta_{ab} + f_{acb}\bar{A}^c_\mu$ is the covariant derivative with $\bar{A}$ inserted instead of A; similarly $\bar{G}^a_{\mu\nu} = G^a_{\mu\nu}(\bar{A})$. As usual the terms which are total derivatives disappear after integrating $\mathcal{L}$. We see that if $\bar{A}$ solves the equation of motion the linear term vanishes, as it should be. If we choose background gauge $\bar{D}^{ac}_\nu B^c_\nu = 0$ the last term of the $O(B^2)$-contribution vanishes. Now we can read the inverse gluon propagator from the terms quadratic in B (from now on we will omit the bars over A, G and D because the unbared objects won't be needed further):

$$(S^{-1})^{ab}_{\mu\nu} = \frac{1}{g^2}(-D^{ac}_\rho D^{cb}_\rho \delta_{\mu\nu} - 2f_{acb}G^c_{\mu\nu}) \quad . \tag{2}$$

We will also omit the $1/g^2$ in front of the propagator which is a result of the rescaling of fields anyway. For further manipulations some abbreviations are useful:

$$\begin{gathered} G_{\mu\nu} = F^c G^c_{\mu\nu} \quad , \quad A_\mu = F^c A^c_\mu \quad , \\ (F^c)_{ab} = i f_{acb} \quad , \quad [F^a, F^b] = i f_{abc} F^c \quad , \quad tr_c F^a F^b = N_c \delta^{ab} \quad , \quad N_c = 3 \quad , \\ (\hat{P}_\mu)^{ab} = i D^{ab}_\mu \quad , \quad \hat{p}_\mu = i\partial_\mu \quad , \quad \hat{P}_\mu = \hat{p}_\mu + A_\mu \quad , \\ \hat{p}_\mu X = [\hat{p}_\mu, X] + X\hat{p}_\mu = i(\partial_\mu X) + X\hat{p}_\mu \quad . \end{gathered} \tag{3}$$

The last equation has only been quoted to show that $\hat{p}$ and $\hat{P}$ will be used in operator sense. F^a are the generators in adjoint representation and f_{abc} are the structure constants of the color gauge group $SU(N_c)$. With these abbreviations we can now write

$$\begin{gathered} S^{-1}_{\mu\nu} = \hat{P}^2 \delta_{\mu\nu} + 2iG_{\mu\nu} = (\hat{p}^2 + \hat{p}\cdot A + A\cdot\hat{p} + A^2)\delta_{\mu\nu} + 2iG_{\mu\nu} = (S_0^{-1} + V)_{\mu\nu} \quad , \\ S^0_{\mu\nu} = \delta_{\mu\nu}/\hat{p}^2 \quad , \quad V_{\mu\nu} = (A^2 + \hat{p}\cdot A + A\cdot\hat{p})\delta_{\mu\nu} + 2iG_{\mu\nu} \quad . \end{gathered} \tag{4}$$

S_0 is the free gluon propator with no background and V can be interpreted as an interaction potential caused by the background. The QCD-Lagrangian in the background 1 can be written in another form more suitable for ordinary perturbation theory:

$$g^2 \mathcal{L}_{QCD} = \frac{1}{4} G^a_{\mu\nu}(B) G^{\mu\nu}_a(B) + \frac{1}{2} B^a_\mu V^{ab}_{\mu\nu} B^b_\nu + f_{abc} f_{aed} B^b_\mu B^c_\nu B^d_v \bar{A}^\mu_e \tag{5}$$

The terms independent and linear in B has been omitted because for a given background the first term is irrelevant and the second can be deleted by shifting B. To determine the weight of a specific background they are of course the most important terms. The Feynman graphs of the additional terms (with g recovered) are depicted in figure . It should be noted that for small coupling constant g the second graph can be treated perturbatively but not the first. So our main concentration lies on the first term.

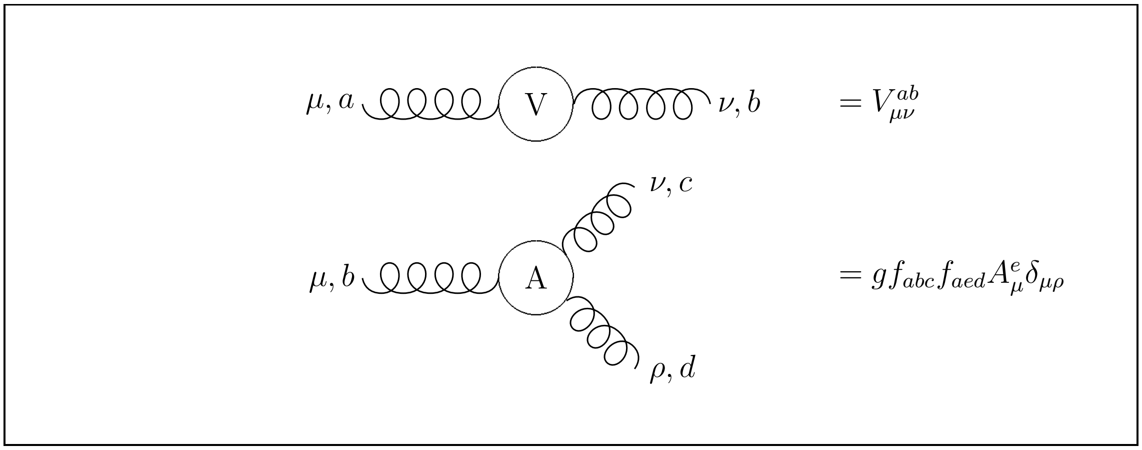

Figure 1: *Feynman rules for gluons in a background*

3. Propagator in Statistical Background

The next step is to use some approximation scheme to calculate the propagator

$$S = (S_0^{-1} + V)^{-1} = S_0(\mathbb{1} + T)^{-1} \quad , \quad T = V S_0 \quad . \tag{6}$$

For large momentum p, S_0 and therefore T are small and we can expand S in powers of T:

$$S = S_0(\mathbb{1} - T + T^2 - T^3 + \ldots) \quad . \tag{7}$$

Note that $S(x, y) = \langle x|S|y\rangle$ is generally not invariant under translations and rotations because the background A_μ and therefore T are not. Actually we are not interested in the propagator for a particular background configuration, but only in the average over all relevant configurations. Here we do not mean the functional integration over quantum fluctuations around the empty vacuum, but fields other than the perturbative $A_\mu\!=\!0$-vacuum which minimize the action $\int \mathcal{L}\, dx$.

$$\overline{S} = S_0(\mathbb{1} - \overline{T} + \overline{T^2} - \overline{T^3} + \ldots) \quad , \tag{8}$$

where the bar denotes averaging over relevant configurations. Details are specified below. If the background is statistically invariant under translations then $\overline{S}$ is translationally invariant and therefore diagonal in momentum space $\overline{S}(p, q) = \langle p|\overline{S}|q\rangle = \overline{S}(p)\eth(p-q)$. If we would evaluate the terms in the series we would get an expansion of $\overline{S}$ in the form

$$\overline{S}(p) = 1/p^2 + c_1/p^4 + c_2/p^6 + \ldots \tag{9}$$

and the pole remains at zero — and gets even worse with higher terms. What we want is $\overline{S}$ in a form like $\overline{S}(p) = (p^2 + M(p)^2)^{-1}$ with $M(p)$ bounded for large and small p and interpreted as momentum dependend gluon mass[2]. So let's invert (8)

$$\overline{S}^{-1} = \hat{p}^2 + M(\hat{p})^2 = (\mathbb{1} - \overline{T} + \overline{T^2} - \overline{T^3} + \ldots)^{-1} S_0^{-1} \tag{10}$$

[2] The " + " in front of M will change to the more familiar " − " when we rotate back from Euclidian space to Minkowski space.

and expand it once again in T. Without averaging this would just be a geometrical series which, expanded, would give the original formula $S^{-1} = S_0^{-1} + V$. With averaging the different terms are now in no relation and expanding and sorting with respect to powers of T yields

$$\begin{aligned}
\overline{S}^{-1} &= (\mathbb{1} + \overline{T} - (\overline{T^2} - \overline{T}^2) + (\overline{T^3} - \overline{T}\,\overline{T^2} - \overline{T^2}\,\overline{T} + \overline{T}^3))S_0^{-1} + O(T^4) \qquad (11)\\
&= S_0^{-1} + \overline{V} - (\overline{VS_0V} - \overline{V}S_0\overline{V}) + \ldots\\
&= S_0^{-1} + M^2, \quad M^2 = M_1^2 - M_2^2 + \ldots, \quad M_1^2 = \overline{V}, \quad M_2^2 = \overline{VS_0V} - \overline{V}S_0\overline{V}.
\end{aligned}$$

In the next section we introduce the concepts of instanton gas calculation to determine M_1, which is actually very simple.

4. First Order Gluon Mass

The classical approximation to a quantum theory in the language of path integrals is to consider only the configurations which minimize the classical Euclidian action S. In QED there is only one minimum, namely $A_\mu = 0$, but in QCD there are other local minima, called instantons:

$$A^a_{I\mu}(x) = O_I^{ab}\eta^I_{b\mu\nu}\frac{(x-z_I)_\nu}{(x-z_I)^2}\frac{2\rho^2}{(x-z_I)^2+\rho^2} \quad , \qquad (12)$$

$$\eta^I_{a\mu\nu} = \overline{\eta}_{a\mu\nu} = \epsilon_{a\mu\nu4} - \frac{1}{2}\epsilon_{abc}\epsilon_{bc\mu\nu} \quad \text{for instantons} \quad ,$$

$$\eta^I_{a\mu\nu} = \eta_{a\mu\nu} = \epsilon_{a\mu\nu4} + \frac{1}{2}\epsilon_{abc}\epsilon_{bc\mu\nu} \quad \text{for anti-instantons} \quad .$$

$A^a_{I\mu}$ is an (anti)instanton of size ρ at position z_I in singular gauge and orientation O_I, where O_I is a rotation matrix in the adjoint representation of $SU(N_c)$. $\eta_{a\mu\nu}$ are the 't Hooft-symbols with some properties given in the appendix of [2]. Whereas the single instanton is an exact solution of the equation of motion $D_\mu^{ab}G^b_{\mu\nu} = 0$, the multi-instanton-field $A = \sum_I A_I$ minimizes S only approximatly, but nevertheless gives a significant contribution to the functional integral. Quantum fluctuations around the instanton renormalize the action $S = 8\pi^2/g^2$ and make small instantons less important because the QCD coupling constant g decreases for small distances [2]. Repulsion between instantons do not allow too large instantons [3], [4, A]. So there is a narrow region of allowed values for the instanton radius and I will use an average radius ρ in my calculations. The second parameter we need is the instanton density n which must be determined experimentally from the gluon condensate [6]. The ratio L_0/ρ is calculated in [3], [4]:

$$n = N/V_4 = 1/L_0^4 = \frac{1}{32\pi^2}\langle G^a_{\mu\nu}G_a^{\mu\nu}\rangle = (200\text{MeV})^4_{exp.} \quad , \quad L_0/\rho = 3.2_{theor.} \qquad (13)$$

Now we plug the N-(anti)instanton-field

$$A = \sum_{I=1}^{N} A_I \tag{14}$$

in our expressions for the gluon propagator (11). To do this we must average some functions $f(A)$ of A like in $\overline{V}$. We will use the instanton gas approximation without interactions between instantons. The whole effects of interactions are summarized in the values of L_0 and ρ. So the position and orientation are equally distributed and independent for different instantons:

$$\overline{f(A)} = \prod_{I=1}^{N} \frac{1}{V_4} \int d^4 z_I \int dO_I \, f(A) \quad , \tag{15}$$

where $\frac{1}{V_4}\int dz_I$ averages over the position of instanton I contained in a large box of volume V_4. $\int dO_I$ is the Haar-measure and averages over the group of orientations

$$\int dO \, 1 = 1 \quad , \quad \overline{O^{ab}} = \int dO \, O^{ab} = 0 \quad ,$$

$$\overline{O^{ab} O^{cd}} = \int dO \, O^{ab} O^{cd} = \frac{1}{N_c^2 - 1} \delta^{ac} \delta^{bd}. \tag{16}$$

We now can easily calculate $\overline{V}_{\mu\nu} = (\overline{A^2} + \hat{p} \cdot \overline{A} + \overline{A} \cdot \hat{p}) \delta_{\mu\nu} + 2i \overline{G}_{\mu\nu}$:

$$\overline{A^a_\mu} = N \overline{A^a_{I\mu}} = N \overline{O^{ab}} \cdot \ldots = 0 \quad \text{and} \quad \overline{G^a_{\mu\nu}} = 0 \tag{17}$$

because $G^a_{\mu\nu}$ is antisymmetric in the Lorentz indices and Lorentz invariant[3] after averaging, which is impossible.

$$\overline{A^2} = \sum_{I=1}^{N} \overline{A_I^2} + \sum_{I \neq J} \overline{A_I A_J} = N \overline{A_I^2} + N(N-1) \overline{A_I}\,\overline{A_J} = N \overline{A_I^2} \quad ,$$

where we have used the fact that different instantons are independent and (17). Inserting now the specific form of the instanton we yield

$$\begin{aligned} (\overline{A^2})_{ab} &= \frac{N}{V_4} \int dz_I \int dO \, i f_{acd} A^c_{I\mu} \, i f_{deb} A^e_{I\mu} \\ &= -\frac{N}{V_4} f_{acd} f_{deb} \overline{O^{cf} O^{eg}} \eta^I_{f\mu\nu} \eta^I_{g\mu\rho} \int d^4 z \frac{z_\nu z_\rho}{z^4} \left(\frac{2\rho^2}{z^2 + \rho^2} \right)^2 \\ &= \frac{12\pi^2 N_c}{N_c^2 - 1} n \rho^2 \delta_{ab} \quad , \quad n = \frac{N}{V_4} = \text{instanton density} \end{aligned} \tag{18}$$

[3] Precisely we should say $O(4)$ invariant because we are in Euclidian space.

where we have used $f_{acd}f_{dcb} = -N_c\delta_{ab}$ and $\eta^I_{\alpha\mu\nu}\eta^I_{\alpha\mu\rho} = 3\delta_{\nu\rho}$. Inserting this into $M_1^2 = \overline{V} = \overline{A^2}$ we get

$$M_1 = \sqrt{\frac{12\pi^2 N_c}{N_c^2-1}(\rho/L_0)L_0^{-1}} \approx 2.1L_0^{-1} = 420\,\mathrm{MeV} \tag{19}$$

Unfortunately M_2 also contains terms which survive the large p limit. To see this one can count the number of $\hat{p}'s$ occuring in M_2^2 which will give us the dominant behaviour of M_2^2 for large p. $\overline{V}S_0\overline{V} = M_1^4/p^2$ vanishes for large p but in principle V contains a $\hat{p}$ in the nominater $(\hat{p}A)$ and $S_0 = 1/\hat{p}^2$ and $\overline{VS_0V}$ could be finite for large p. To be more definite consider

$$VS_0V = (pA + Ap)S_0(pA + Ap) + \text{other terms} = 4A_\mu\frac{p_\mu p_\nu}{p^2}A_\nu + \ldots$$

where we have used $[p, A_I] = i\partial_\mu A_I^\mu = 0$.

$$\langle x|VS_0V|y\rangle = 4A_\mu(x)\langle x|\frac{p_\mu p_\nu}{p^2}|y\rangle A_\nu(y) + \ldots$$

From $\delta_{\mu\nu}\langle x|p_\mu p_\nu/p^2|y\rangle = \langle x|y\rangle = \delta(x-y)$we can conclude that$\langle x|p_\mu p_\nu/p^2|y\rangle = \frac{1}{4}\delta_{\mu\nu}\delta(x-y)$ + traceless terms.

$$\langle x|VS_0V|y\rangle = A_\mu(x)A_\mu(x)\delta(x-y) + \ldots = \langle x|A^2|y\rangle + \ldots$$

So $M_2^2 = \overline{A^2} + \ldots = M_1^2 + \ldots$ contains a term which fully cancels M_1^2 calculated above. In the next section we use an approximation which overcomes this problem but actually restrict ourself in a calculation for small p.

5. Propagator in statistical Background (Expansion in Instanton Density)

In this section we will make an expansion of the gluon propagator in the instanton density n which will be valid for all euclidian momenta p especially for small p.

The average of an arbitrary power of the 1-instanton field is proportional to the inverse volume $\overline{A_I^n} \sim \frac{1}{V}$ for $n \geq 1$. So, for example, the expansion of the square of the multi-instanton configuration in the instanton density is

$$\overline{A^2} = \sum_{I=1}^{N}\overline{A^2} + \underbrace{N\overline{A_I}}_{O(n)}\underbrace{(N-1)\overline{A_I}}_{O(n)} = \underbrace{N\overline{A_I^2}}_{O(n)} + O(n^2) \tag{20}$$

and more general

$$\overline{A^n} = N\overline{A_I^n} + O(n^2) \quad , \quad \overline{V^n} = N\overline{V_I^n} + O(n^2) \quad \text{for} \quad n \geq 1, \tag{21}$$

where $V_I = V(A_I)$ defined in (4) with A replaced by A_I and similar for T. Let us now sum up all terms in (11) linear in n:

$$\begin{aligned}\overline{S}^{-1} &= (1\!\!1 + \overline{T} - \overline{T^2} + \overline{T^3} - \ldots)S_0^{-1} + O(n^2) \\ &= (1\!\!1 + N(\overline{T_I} - \overline{T_I^2} + \overline{T_I^3} - \ldots))S_0^{-1} + O(n^2) \\ &= (1\!\!1 + N\overline{T_{eff}})S_0^{-1} + O(n^2) = S_0^{-1} + N\overline{V_{eff}} + O(n^2) \end{aligned} \tag{22}$$

$$\begin{aligned} T_{eff} &= T_I - T_I^2 + T_I^3 - \ldots = T_I - T_I(T_I - T_I^2 + \ldots) = T_I - T_I T_{eff} \\ V_{eff} &= V_I - V_I S_0 V_{eff} \quad \Longrightarrow \quad V_{eff} = S_0^{-1}(S_0 - S_I)S_0^{-1} \quad \text{with} \qquad (23) \\ S_I^{-1} &= S_0^{-1} + V_I \quad \text{is the propagator in the 1-instanton background.} \qquad (24) \end{aligned}$$

More generally we can make a cluster expansion of an arbitrary function of A in the following form

$$\overline{f(A_1 + \ldots + A_N)} = \overline{f(A)} = \sum_{l=0}^{N} \binom{N}{l} \overline{\sum_{k=0..l} (-)^{l-k} \binom{l}{k} f(A_1 + \ldots + A_k)} = \tag{25}$$

$$= \underbrace{f(0)}_{O(1)} + \underbrace{N\overline{f(A_1) - f(0)}}_{O(n)} + \frac{1}{2}\underbrace{N(N-1)\overline{f(A_1 + A_2) - 2f(A_2) + f(0)}}_{O(n^2)} + O(n^3)$$

where the first line is an identity even without averaging. Inserting a Taylor expansion for f in the second line and using the indistinguishability of different instantons one can see that all monomials in the k-th term contain more or equal than k different instantons. So the average will factorize in k factors each proportional to n and therefore the k-th term is indeed proportional to n^k. It is easy to generalize (25) for two or more species of fields. If A is a field of N_I instantons *and* $N_{\bar{I}}$ antiinstantons we get to first order in n

$$\overline{f(A)} = f(0) + N_I\overline{f(A_I) - f(0)} + N_{\bar{I}}\overline{f(A_{\bar{I}}) - f(0)} + O(n^2). \tag{26}$$

If we insert the propagator S in (26) we get

$$\overline{S} = \overline{S(A)} = S(0) + N\overline{S(A_I) - S(0)} + O(n^2) = S_0 + N\overline{S_I - S_0} + O(n^2) \tag{27}$$

which is after inversion up to $O(n^2)$ just (22).

6. QCD Propagators

In section 5 we have seen that it is enough to know the 1-instanton propagator to calculate $\overline{S}$ to first order in n. Luckily this propagator is known, unfortunatly it is a bit lengthy and suffers from a divergency.

The gluon propagator $S^{ab}_{I\mu\nu}$ with spin $S = 1$ in adjoint color[4] representation ($C = 1$) can be constructed out of the ghost propagator Δ^{ab}_I ($S = 0, C = 1$) which is explicitly

[4] Often called isospin [9]

known in the 1-instanton background. The general formulas how to construct a propagator of given spin S out of the corresponding scalar propagator with same color C in a selfdual background are shown below. They are derived and more thoroughly discussed in [9].

Notations:

$$A_\mu = T^a A^a_\mu \quad , \quad [T^a, T^b] = i\epsilon_{abc} T^c \quad , \quad T^a = \text{generator of} \quad SU(2)_c$$

$$T^a = \left\{ \begin{array}{rll} 0 & \text{in scalar} & (C=0) \\ \tau^a/2 & \text{in fundamental} & (C=\frac{1}{2}) \\ i\epsilon_{\cdot a \cdot} & \text{in adjoint} & (C=1) \end{array} \right\} \text{representation} \tag{28}$$

$$P_\mu = p_\mu + A_\mu \quad , \quad p_\mu = i\partial_\mu \quad , \quad \tilde{G}_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\rho\sigma} G_{\rho\sigma} \quad , \quad \{\gamma_\mu, \gamma_\nu\} = 2\delta_{\mu\nu}$$

Spin 0 *propagator* $\tilde{\Delta}$*:*

$$\tilde{\Delta}^{-1} = P^2 \quad \Longrightarrow \quad \tilde{\Delta} = P^{-2} \tag{29}$$

Spin $\frac{1}{2}$ *propagator* S*: (not used but only stated for completeness)*

$$S^{-1} = \not{P} = \gamma_\mu P^\mu \quad \Longrightarrow \quad S = \not{P}\tilde{\Delta}\frac{1+\gamma_5}{2} + \tilde{\Delta}\not{P}\frac{1-\gamma_5}{2} \quad \text{for} \quad G_{\mu\nu} = \tilde{G}_{\mu\nu} \tag{30}$$

Spin 1 *Propagator* S*:*

$$S^{-1}_{\mu\nu} = P^2\tilde{\Delta}_{\mu\nu} + 2iG_{\mu\nu} - (1-\frac{1}{\xi})P_\mu P_\nu \quad \Longrightarrow$$

$$S_{\mu\nu} = q_{\mu\nu\rho\sigma} P_\rho \tilde{\Delta}^2 P_\sigma - (1-\xi) P_\mu \tilde{\Delta}^2 P_\nu \quad \text{for} \quad G_{\mu\nu} = \tilde{G}_{\mu\nu}, \tag{31}$$
$$q_{\mu\nu\rho\sigma} = \delta_{\mu\nu}\delta_{\rho\sigma} + \delta_{\mu\rho}\delta_{\nu\sigma} - \delta_{\mu\sigma}\delta_{\nu\rho} + \epsilon_{\mu\nu\rho\sigma} \quad .$$

There are some comments in order. The above formulas are valid for an arbitrary color representation, we will need them only for $C = 1$. For $S \neq 1$ there are zeromodes and the propagator is only the inverse of the kernel in a subspace orthogonal to the zeromodes. The implications and problems will be discussed in section 8 when they show up explicitly. The Spin 1 kernel is the quadratic term of the QCD-Langrangian (1) with gauge fixing term $\frac{1}{2\xi}(D^{ab}_\mu B^b_\mu)^2$ in slight generalisation to the $\xi = 1$ case considered previously.

With

$$\Pi(x) = 1 + \frac{\rho^2}{x^2} \quad , \quad F(x,y) = 1 + \rho^2 \frac{(\tau x)}{x^2}\frac{(\tau^\dagger y)}{y^2} \quad , \tag{32}$$

$$\tau_\mu = (\vec{\tau}, i) \quad , \quad \tau^\dagger_\mu = (\vec{\tau}, -i) \quad , \quad \tau_\mu \tau^\dagger_\nu = \delta_{\mu\nu} + i\bar{\eta}_{a\mu\nu}\tau_a$$

we can write the ghost propagator for 1 instanton in the form [9]

$$
\begin{aligned}
\Delta_I^{ab}(x,y) &= \frac{\frac{1}{2}\text{tr}\ \tau_a F(x,y)\tau_b F(y,x)}{4\pi^2(x-y)^2\Pi(x)\Pi(y)} = \\
&= \frac{\delta_{ab}}{4\pi^2(x-y)^2} - \frac{\rho^2\delta_{ab}}{4\pi^2(x^2+\rho^2)(y^2+\rho^2)} + \frac{2\rho^2\epsilon_{abc}\bar{\eta}_{c\mu\nu}x_\mu y_\nu}{4\pi^2(x-y)^2(x^2+\rho^2)(y^2+\rho^2)} \\
&\quad + \frac{2\rho^4(((xy)^2-x^2y^2)\delta_{ab}+\epsilon_{abc}\bar{\eta}_{c\mu\nu}x_\mu y_\nu+\bar{\eta}_{a\mu\nu}\bar{\eta}_{b\rho\sigma}x_\mu y_\nu x_\rho y_\sigma}{4\pi^2(x-y)^2(x^2+\rho^2)x^2(y^2+\rho^2)y^2} \qquad (33)
\end{aligned}
$$

For simplicity we have placed the instanton at the origin $z_I = 0$ in standard orientation $O^{ab} = \delta^{ab}$. It is possible to write down the gluon propagator $S_{I\mu\nu}^{ab}$ explicitly in which we are finally interested in using (31) but the expression will be rather lengthy. While Δ_I^{ab} can be averaged and represented in momentum space exactly with the help of modified Besselfunctions this seems not to be possible for $S_{I\mu\nu}^{ab}$ and we have to expand $\overline{S}$ for large and/or small momenta (or work much harder to solve the complicated integrals in terms of special functions). So we will never use the full expression for S.

7. Propagators for small momentum

The calculation simplifies significantly if we expand $\overline{\Delta}(p)$ or $\overline{S}(p)$ for small momentum. Because p always occurs in the dimensionless quantity $(p\rho)$ the lowest order in p can be found by keeping only terms of lowest order in ρ or differently stated: Small p corresponds to large x and for $x \gg \rho$ ρ is neglegible. To warm up let us start with Δ from (33):

$$\Delta_I^{ab} = \Delta_0^{ab} - \rho^2 W^{ab} + O(\rho^4) \qquad (34)$$

$$\Delta_0^{ab}(x,y) = \frac{\delta_{ab}}{4\pi^2(x-y)^2}$$

$$W^{ab}(x,y) = \frac{\delta_{ab}}{4\pi^2x^2y^2} + \frac{2\epsilon_{abc}\bar{\eta}_{c\mu\nu}x_\mu y_\nu}{4\pi^2(x-y)^2x^2y^2}$$

$\Delta_0(p) = 1/p^2$ is the free ($A_\mu^a \equiv 0$) ghost propagator. After reintroducing the instanton position z we can now average $\Delta_0 - \Delta_I$. The ϵ-term in W will be killed by averaging over the instanton orientation:

$$\langle x|\overline{\Delta_0^{ab} - \Delta_I^{ab}}|y\rangle = 4\pi^2\delta_{ab}\rho^2\frac{1}{V_4}\int\frac{d^4z}{4\pi^2(x-z)^2 4\pi^2(z-y)^2} \quad , \qquad (35)$$

The last integral is infrared diverent but noticing that $1/4\pi^2(x-y)^2$ is the fourier transformation of $1/p^2$ we can rewrite the above expression in the form

$$V_4\langle x|\overline{\Delta_0 - \Delta_I}|y\rangle = 4\pi^2\rho^2\int d^4z\langle x|\frac{1}{p^2}|z\rangle\langle z|\frac{1}{p^2}|y\rangle = 4\pi^2\rho^2\langle x|\frac{1}{p^4}|y\rangle$$

The divergence is now hidden in the fact that the coordinate representation of p^{-4} does not exist. In fact we are not interested in the coordinate representation but in the momentum representation and so the divergence is spurious. Inserting Δ in (23) we get

$$\overline{V}_{eff}^{ghost} = \Delta_0^{-1}\ \overline{\Delta_0 - \Delta_I}\ \Delta_0^{-1} = p^2 \frac{4\pi^2\rho^2}{V_4} \frac{1}{p^4} p^2 = \frac{1}{V_4} 4\pi^2\rho^2 = \rho^2 p^4 \overline{W} \tag{36}$$

and finally

$$M_{ghost}^2(p=0) = N\overline{V}_{eff} = 4\pi^2\rho^2 n \approx (420\mathrm{MeV})^2$$

which has to be inserted in the ghost propagator $\overline{\Delta}_{ab} = \delta_{ab}(p^2 + M_{ghost}^2(p))^{-1}$.

What can we learn from this? At first $M \neq 0$. A scalar particle moving in the instanton background gets a dynamical soft mass. The pole at $p = 0$ has vanished. Secondly $M(p \to 0)$ is finite. reasonable to think that M is bounded (has no poles) at least in the Euclidian region $p^2 > 0$. From an exact averaging of (33) one can see that M is finit for all momenta [10]. This was our motivation to prefer the representation (11) instead of (9). Let me make a last comment on this inversion. In ordinary perturbation theory one calculates the one loop self energy, sums up the series of one loop two consecutive loops and so on to bring the selfenergy in the denominator of the propagator which actually is equivalent to a simple inversion and hopes to improve the result with this partial resummation. The arguments for doing this are not better or weaker than here - they are essentially the same.

Let us now calculate the gluon mass at zero momentum along the same lines. To do this we must insert (34) into (31). Expanding also P_μ up to order ρ^2

$$P_\mu = p_\mu + \rho^2 \breve{A}_\mu + O(\rho^4) \quad , \quad \breve{A}_\mu = \frac{2F_a\bar{\eta}_{a\mu\nu}x_\nu}{x^4}$$

we get

$$P_\mu \Delta_I^2 P_\nu = p_\mu \Delta_0 p_\nu - \rho^2 p_\mu(\Delta_0 W + W\Delta_0)p_\nu + \rho^2(p_\mu \Delta_0 \breve{A}_\nu + \breve{A}_\mu \Delta_0 p_\nu) + O(\rho^4)$$

The free gluon propagator in R_ξ-gauge is well known or can be obtained from (31) setting $G_{\mu\nu} = 0$:

$$S_{\mu\nu}^0 = q_{\mu\nu\rho\sigma} p_\rho \Delta_0^2 p_\sigma - (1-\xi) p_\mu \Delta_0^2 p_\nu = \frac{1}{p^2}(\delta_{\mu\nu} - (1-\xi)\frac{p_\mu p_\nu}{p^2})$$

$$S_{\mu\nu}^0 - S_{\mu\nu}^I = \rho^2[q_{\mu\nu\rho\sigma} p_\rho (\Delta_0 W + W\Delta_0) p_\sigma - (1-\xi) p_\mu (\Delta_0 W + W\Delta_0) p_\mu]$$

$$\overline{S_{\mu\nu}^0 - S_{\mu\nu}^I} = 2\rho^2 \overline{W}(p^2\delta_{\mu\nu} - (1-\xi)p_\mu p_\nu) = 2\rho^2 p^2 \overline{W} S_{\mu\nu}^0$$

where we have used in the last line that p comutes with averaged quantities like $\overline{W}$.

$$\overline{V}_{eff}^{gluon} = S_0^{-1}\overline{S_0 - S_I} S_0^{-1} = 2\rho^2 p^2 S_0^{-1}\overline{W}$$

$$\overline{S}^{-1} = S_0^{-1} + N\overline{V_{eff}^{gluon}} = S_0^{-1}(\mathbf{1} + 2M_{ghost}^2(0)/p^2)$$

$$\overline{S} = \frac{p^2 S_0}{p^2 + M_{ghost}^2(0)} = \frac{\delta_{\mu\nu} - (1-\xi)\frac{p_\mu p_\nu}{p^2}}{p^2 + M_{gluon}^2(p^2)} \quad \text{with}$$

$$M_{gluon}^2(p^2) = 2M_{ghost}^2(0) + O(p^2)$$

Maybe the relation $M_{gluon} \approx 2M_{ghost}$ is valid even for larger p. Till now we have dealt with 2-color QCD $N_c = 2$. The extension of our result to the real world needs a thorough adaption of all formulas in this work and [9] to $N_c = 3$. After averaging there will appear several factors $N_c/(N_c^2 - 1)$ like in (19) and $3/(N_c^2 - 1)$ compared to 2/3 and 1 in the $N_c = 2$-case. So most probable M_{gluon}^2 has to be multiplied with a factor between 3/8 and 1/2 yielding a gluon mass of approximately 400MeV for $N_c = 3$.

8. Conclusions and further developments

In our whole calculation we have ignored the zeromodes. All gluon field fluctuations must be orthogonal to these which is achieved by using a gluon propagator orthogonal to the zeromodes or otherwise stated by subtracting out from (31) the projections on the zeromode subspace. This procedure also deletes a divergence comming from the second term in (33) squared. There are some further problems to be solved because the scalar product of the zeromodes with the propagator does not exist [9]. But all this concerns a term which is proportional to ρ^4 which was irrelevant in our zero momentum approximation. Also the principle mechanism of mass generation does not depend on the zeromodes which can be seen from the ghost propagator because in the spin 0 case there are no zeromodes. This should be contrasted with the fermionic case where the zeromode are the main ingredient for mass generation and the so called zeromode approximation simplifies calculations enormously.

The next step may be the calculation of some gauge invariant correlation function like a glueball correlator or topological susceptibility. As usual this involves products and integrals over propagators but with the difference that we have to use (31) instead of the simple free propagator. Without further simplification this may cause a headache.

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