

Family Structure from Periodic Solutions of an Improved Gap Equation

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Abstract

Fermion mass models usually contain a horizontal symmetry and therefore fail to predict the exponential mass spectrum of the Standard Model in a natural way. In dynamical symmetry breaking there are different concepts to introduce a fermion mass spectrum, which automatically has the desired hierarchy. In constructing a specific model we show that in some modified gap equations periodic solutions with several fermion poles appear. The stability of these excitations and the application of this toy model are discussed. The mass ratios turn out to be approximately e^π and $e^{2\pi}$. Thus the model explains the large ratios of fermion masses between successive generations in the Standard Model without introducing large or small numbers by hand.

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1 Introduction

Although the Standard Model describes the low energy physics with relatively few parameters, model building tends to increase the number of particles and the degrees of freedom with a low reduction of the number of parameters. That means either that the low energy physics can only be understood at the Planck scale or that a complex dynamical structure is responsible for the pattern of masses and couplings. The family duplication is the central problem and one will not be able to find the origin of mass generation without facing that problem. There were a lot of attempts to understand the mass splittings and the mixing angles in the CKM-matrix [1]-[5]. One possibility is to study different textures as it was done in the classical papers by B.Stech and H.Fritzsch [1]. They are followed by a lot of papers partly explaining the high top mass [2]. Some authors have searched for infrared fixed points of the running couplings to determine the masses and mixing angles [3]. In constructing underlying models predicting the parameters from first principles one has studied composite models of quarks and leptons [4] or has established a horizontal symmetry to explain the different generations [5].

In all these models a mass spectrum

$$m_k \approx m_1 \cdot e^{(k-1)\alpha}, \quad \text{Generation } k = 1, 2, 3 \quad (1)$$

as it is realized in the Standard Model is hard to describe¹ (see fig.1 [6]).

E.g. a horizontal symmetry must be broken drastically. Hence it seems more reasonable to start with a theory which predicts an exponential mass spectrum in a natural way from the very beginning.

Dynamical symmetry breaking models are excellent candidates. There are two possibilities to generate excitations. One may interpret either the different solutions of a gap equation or the different propagator poles of one solution as particles of higher generations. In the first case [7] each solution of the gap equation represents another minimum of the effective action. The generation number of the particle depends on the vacuum, where it is living. Hence the excited states must move in vacuum bubbles, which unfortunately are rather unstable [8]. Nevertheless the mass matrices and therefore the CKM matrix might in principle be determined by calculating the instanton transitions between different vacua and one gets the Standard Model as an effective description of that scenario. Hence the

¹ The masses at a fixed scale (e.g. 1GeV) must be compared rather than the on-shell masses as long as the running of the quark masses is not involved in the model.

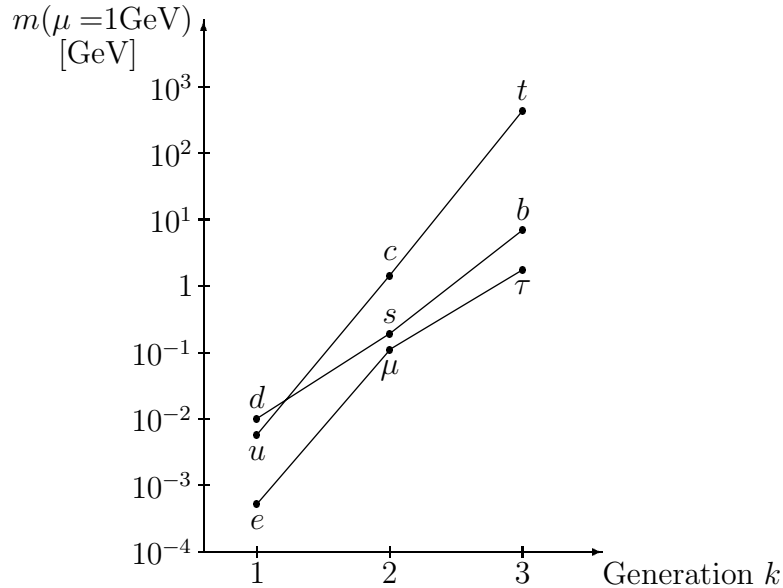


Figure 1 *Fermion mass spectrum.*

model might be phenomenological acceptable, if the problem of the bubble instability can be solved.

In this paper we will concentrate on the second case, where we are interested in gap equations with periodic solutions. Here the fermion propagator $S(p^2) = \frac{i}{\not{p} - B(p^2)}$ with the dynamically generated mass function $B(p^2)$ should have several poles m_k^2 with $m_k^2 - B(m_k^2)^2 = 0$, where m_k is the corresponding mass spectrum. This is rather non-trivial since a real continuous periodic function $p^2 - B(p^2)^2$ cuts the p^2 -axis both from above and below, which alternately leads to poles with positive and negative residue corresponding to particles and ghosts. A ghost pole in $S(p^2)$ is avoided, if $B(p^2)^2$ admits a pole or an imaginary part in the p^2 -region between two particle states. An imaginary part above a fermion pole appears anyway. It describes the decay of the off-shell fermion. But it is unusual that this imaginary part disappears again at the next fermion pole.

As shown in section 2 a periodic fermion mass function cannot be found in the simple ladder approximation. One has to improve the corresponding gap equation by introducing a running coupling to find periodic solutions (section 3). The different poles of the propagator yield mass ratios consistent with the Standard Model mass spectrum.

In section 4 we show that the different fermion poles really can be interpreted as different particle states with own propagators. At the tree level one cannot decide whether there

is one propagator with several poles or several propagators.

In section 5 an effective description of our model is given, where one naively expects flavor changing transitions. But using only gauge invariance we will show that the particle spectrum is stable.

Section 6 generalizes the discussion of the previous sections to more realistic models containing a fermion isodoublet per generation.

The model therefore naturally contains the essential properties of the mass generation and the mass spectrum of the Standard Model.

2 The gap equation in the ladder approximation

We study a toy model of one fermion field coupled minimally to a boson field. In section 6 the fermion field will be transferred to the Standard Model fermions and the boson field to Standard Model gauge bosons or to a hidden sector of strongly coupled vector fields. A fundamental Higgs boson is absent. The dynamical mass generation should rather be compared to a QCD-like scenario. The fermion selfenergy can be determined by the Schwinger–Dyson selfconsistency equation

$$\text{---} \bullet \text{---} = \text{---} \bullet \text{---} \text{---} \bullet \text{---} \quad (2)$$

where on the right hand side the full fermion propagator and the exact boson–fermion vertex appears.

Now the standard procedure to get a closed selfconsistency equation is to replace the exact vertex by the tree level vertex. There are many justifications for this approximation. The most important point is that it is a consistent approximation. It respects all symmetries of the theory. Especially the Ward identities are preserved in the case of a gauge theory since the vertex can be chosen in an appropriate manner. Further in a non-abelian gauge theory with N_c colors this approximation is identical to the leading $1/N_c$ term. For more details see [9].

Our starting point will therefore be the gap equation

$$iS^{-1}(p) - \not{p} = \frac{1}{i} \int \frac{d^4k}{(2\pi)^4} \Gamma^a S(k) \Gamma_a D(p-k) \quad (3)$$

where $S(p) = i(A(p^2)\not{p} - B(p^2))^{-1}$ is the fermion propagator, $D(q)$ the propagator of the

interacting boson and Γ_a the boson–fermion vertex proportional to some coupling g . The index a runs over Lorentz and other group indices. The wavefunction $A(p^2)$ does not play any important role in the game. Hence we set $A(p^2) \equiv 1$ for simplicity². We get:

$$B(p^2) = i \int \frac{d^4k}{(2\pi)^4} \frac{\Gamma^a \Gamma_a B(k^2)}{k^2 - B(k^2)^2} \frac{1}{(p-k)^2} \quad (4)$$

or for Euclidean momenta:

$$B(-p^2) = -C \int \frac{d^4k}{\pi^2} \frac{B(-k^2)}{k^2 + B(-k^2)^2} \frac{1}{(p-k)^2} \quad (5)$$

where $C = \Gamma^a \Gamma_a / (4\pi)^2$. Since $\square \frac{1}{q^2} = -4\pi^2 \delta^4(q)$, we get:

$$\frac{1}{4} \square B(-p^2) \equiv \frac{1}{p^2} \frac{d}{dp^2} \left(p^4 \frac{d}{dp^2} B(-p^2) \right) = C \frac{B(-p^2)}{p^2 + B(-p^2)^2}. \quad (6)$$

There are two boundary conditions [10]

$$\left[B(-p^2) - \frac{d}{dp^2} (p^2 B(-p^2)) \right]_{p^2=0} = 0 \quad \text{and} \quad \left. \frac{d}{dp^2} (p^2 B(-p^2)) \right|_{p^2=\Lambda^2} = 0 \quad (7)$$

(Λ is the momentum space cutoff.) which are fulfilled, if $B(0)$ is finite and C is above the critical value $1/4$ [11].

We simplify and rotate back to Minkowski space:

$$\left(\frac{d}{dp^2} \right)^2 (p^2 B(p^2)) = C \frac{B(p^2)}{p^2 - B(p^2)^2} \quad (8)$$

This scale independent differential equation can be written in an autonomous form replacing $p^2 = e^t$ and $B(p^2) = e^{t/2} y(t)$:

$$\ddot{y} + 2\dot{y} + \frac{3}{4}y - \frac{Cy}{1-y^2} = 0. \quad (9)$$

It looks like an equation of motion of a classical mass point moving with friction $2\dot{y}$ in a potential

$$U(y) = \frac{3}{8}y^2 + \frac{C}{2} \ln |y^2 - 1|. \quad (10)$$

A finite³ $B(0)$ corresponds to $y(-\infty) = \infty$ and from $B(p^2) \rightarrow 0$ we find $y(t) \rightarrow 0$, which should happen far beyond the mass generation scale i.e. in the limit $p \rightarrow \infty$. Hence the

²One can also get rid of $A(p^2)$ by using the Landau gauge, if $D(q)$ is a gauge boson propagator.

³ $B(0)$ can be scaled by shifting t due to scale invariance. The solution is therefore independent of any boundary conditions.

system must pass the peak of the potential at $y = 1$. For $C < 0$, as it is the case in a vector theory, $U(1) = +\infty$ and y will never cross, so that $y > 1$ and $B(p^2)^2 > p^2$ for all p^2 . Thus the fermion has no mass pole, which was first mentioned by Fukuda and Kugo [12]. This problem is connected to the infrared problem of QED, but it is hard to solve in the Schwinger–Dyson picture in an honest way. This problem is avoided when replacing $B(p^2)^2$ in the denominator of eq.(8) by a constant. Unfortunately this very popular approximation [10] destroys much of the nice structure, which we will explore in the sequel.

For a positive C , e.g. in an axial vector theory, one does not run into difficulties and we will concentrate on this case. Our calculation can also be performed for negative C , but it needs more effort to deal with the infrared problem.

The line $y = 1$ corresponds to the line $p^2 - B(p^2)^2 = 0$. All t 's with $y(t) = 1$ are fermion poles which we are interested in. We therefore define $u := y - 1$ to discuss the pole behaviour at the origin. We get:

$$\ddot{u} + 2\dot{u} + \frac{3}{4}(u + 1) + \frac{C}{2} \left(\frac{1}{u} + \frac{1}{u + 2} \right) = 0 \quad (11)$$

The force vectorfield of the corresponding mechanical problem is shown in fig.2 and a typical solution of this equation is plotted in fig.3 and fig.4. The solution for u starts at real infinity. At the origin ($u = 0$) the fermion pole provides a kick and leads to an imaginary part. For $t \rightarrow \infty$ u tends to the fixed point -1, which corresponds to $B(p^2) \rightarrow 0$. We only get one fermion pole.

Away from $u = 0$ the solution follows roughly the vectorfield in an adiabatic way indicating that the acceleration \ddot{u} is small. Neglecting \ddot{u} , an analytic solution of eq.(11) can be obtained. The behaviour near the pole can also be studied analytically. Near $u = 0$ the potential gets large and negative. The differential equation (11) is dominated by the acceleration \ddot{u} and the force $1/u$. Near the pole the solution is therefore described by the equation $\ddot{u} + C/2u = 0$ which can be solved exactly in terms of the error function. Starting at positive u_0 the curve reaches $u = 0$. Beyond this point the solution acquires a positive/negative imaginary part if we choose $u_0 \pm i\varepsilon$ as our starting point. Note that there is no kink at the origin. The curve bends smoothly into the complex plane.

As shown in this approximation we do not find any periodicity of the solution. The curve in fig.3 does not bend back to cross the origin again.

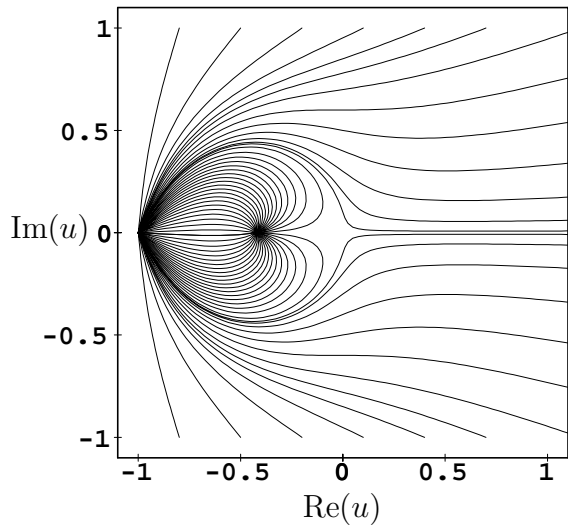


Figure 2 Phaseportrait of the vectorfield in the complex plane for $C = 0.5$.

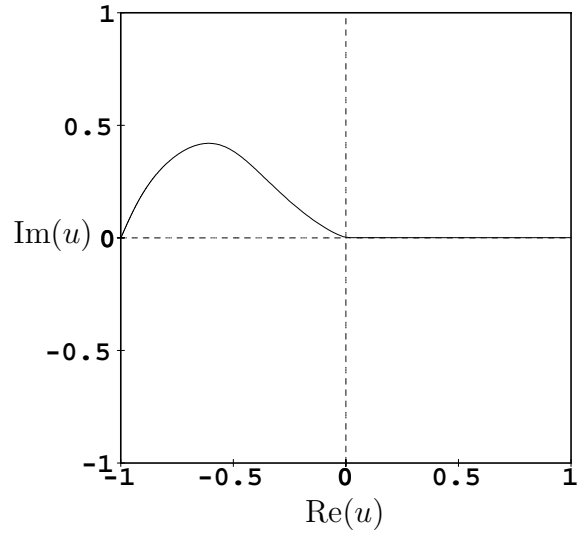


Figure 3 Solution for $C = 0.5$.

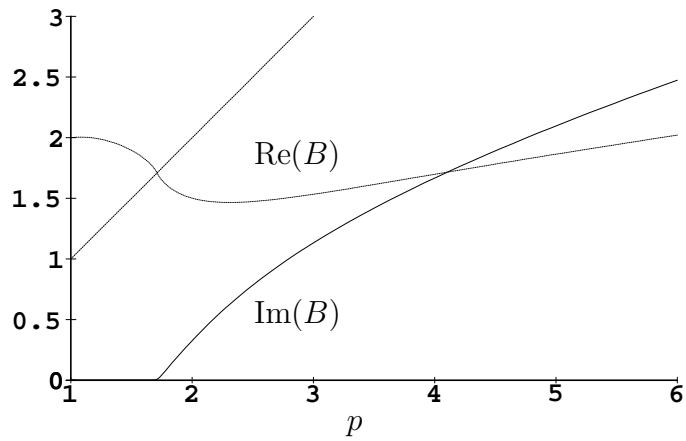


Figure 4 $B(p^2)$ for $C = 0.5$.

3 Periodic solutions for gap equations with a running coupling

The situation changes if we introduce a running coupling constant. The simplest way to do that is to replace the constant factor C in eq.(8) by a momentum dependent function

$\mathcal{C}(p^2)$. It effectively describes the solution of a Schwinger–Dyson equation for the vertex function. But a running coupling destroys the scale invariance of our differential equation. Two effects must be separated: One is a smooth logarithmic momentum dependence connected with the anomalous dimension of the coupling, which we will ignore until the discussion of the number of generations at the end of this section. The remaining non-anomalous coupling \mathcal{C} must be a function of dimensionless quantities and usually describe mass threshold effects. In our case it must therefore depend non-trivially on $p^2/B(p^2)^2$ caused by the dynamically generated masses. Going along the steps in section 2 we end up with

$$\ddot{u} + 2\dot{u} + \frac{3}{4}(u+1) + \frac{\mathcal{C}(u)}{2} \left(\frac{1}{u} + \frac{1}{u+2} \right) = 0 \quad (12)$$

which can be parametrized by a Taylor expansion up to the quadratic term:

$$\ddot{u} + 2\dot{u} + \frac{3}{4}(u+1) + \frac{\mathcal{C}(0)}{2} \left(\frac{1}{u} + \frac{1}{u+2} \right) + \alpha + \beta u + \gamma u^2 = 0 \quad (13)$$

Now for a wide range of parameters one gets periodic solutions as in fig.5, where we have chosen $\mathcal{C}(0) = 1.5$, $\alpha = -1.5$, $\beta = -1.0$ and $\gamma = 1.0$. The insertion of this solution into the gap equation for the vertex function would determine in turn the coupling $\mathcal{C}(u)$ which should be consistent with our ansatz. This is a difficult task, which will not be addressed in this work. Instead we directly want to study the phenomenological consequences of the solutions.

The behaviour near $u = 0$ is unchanged and the analysis of the last section can be adopted. But for large u the solution now bends back to the real positive axis allowing periodic solutions. In each cycle $u = 0$ is passed from the right to the left in the same way. Therefore all poles have the same residue. Positivity of all residues guarantees that all poles correspond to physical particles (not ghosts). The poles of the propagator occur at $u(0) = u(T) = u(2T) = \dots = 0$, where T is the period of the solution. For the example given above $T = 12.1$. From the stability condition of the trajectory around the fixed point u_F of the differential equation (13) one can roughly estimate⁴:

$$T \approx 2\pi L \cdot \left| \frac{\text{Re}(u_F)}{\text{Im}(u_F)} \right| \left(1 + \sqrt{1 - 2\gamma \text{Re}(u_F) |u_F|^2} \right)^{-1} \approx 2\pi L \quad (14)$$

for $\text{Im}(u_F) \approx \text{Re}(u_F)$ and generic values for γ . L is the number of loops of the trajectory around the fixed points. In fig.5 there are 2 fixed points inside the trajectory and hence

⁴This can be calculated by expanding eq.(13) around u_F and demanding a stable circle as the orbit of the system around that fixed point.

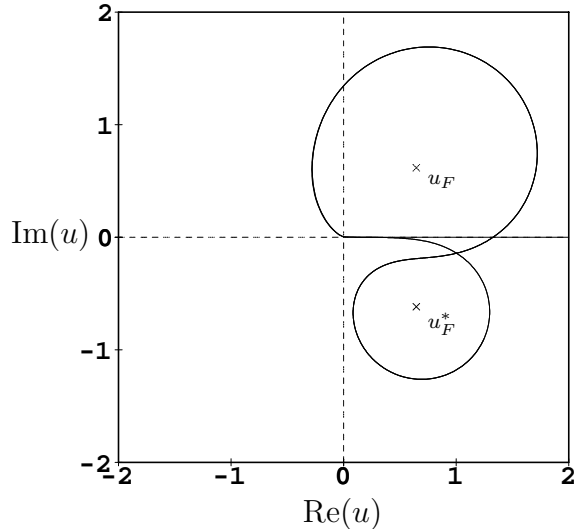


Figure 5 Solution for $\mathcal{C}(0) = 1.5$, $\alpha = -1.5$, $\beta = -1.0$ and $\gamma = 1.0$. There are two fixed points $u_F^{(*)} = 0.646 \pm 0.617i$.

$L = 2$. Curves surrounding only one fixed point are possible, too (see fig.8). However the coupling $\mathcal{C}(u)$ needs a soft fine tuning in that case.

Now we get infinitely many elementary fermionic excitations with an exponential mass spectrum

$$m_n = m_0 e^{nT/2}, \quad n \in \mathbb{Z} \quad (15)$$

Scale invariance of eq.(3) by an arbitrary factor λ is broken dynamically down to a discrete subgroup generated by a scaling with $e^{T/2}$, which transforms a state of mass m_n into a state of mass m_{n+1} .

One can try to write down an effective action with an elementary field for each n . Due to the fact that the residue of all poles is the same, the fermions will be coupled in a universal manner to the gauge boson. Absence of flavor changing currents and the stability of the spectrum will be discussed in section 5. Many features of the family structure of the standard model have so far be reproduced in a qualitative manner. Even the large mass ratios of the order of

$$m_{n+1}/m_n = e^{T/2} \approx 16 \dots 300 \quad (16)$$

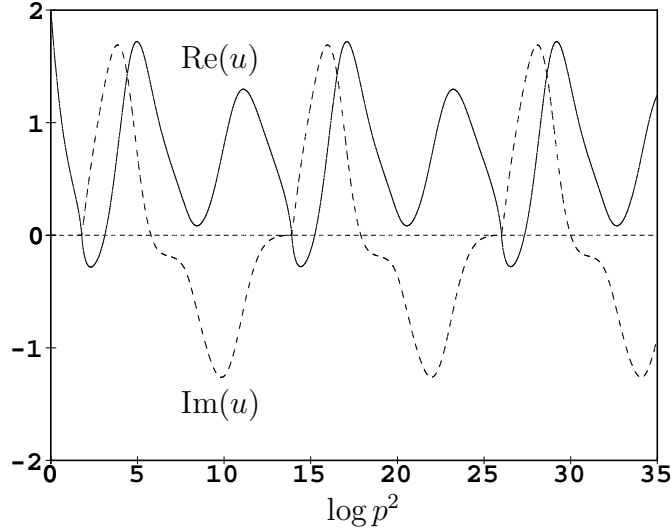


Figure 6 Real and imaginary part for $\mathcal{C}(0) = 1.5$, $\alpha = -1.5$, $\beta = -1.0$ and $\gamma = 1.0$.

with [6]

$$T_{\mu-\tau} = 5.6, \quad T_{d-s} = 6.0, \quad T_{s-b} = 7.1, \quad T_{e-\mu} = 10.7, \quad T_{u-c} = 11.2, \quad T_{c-t} = 11.3 \quad (17)$$

can be understood in our model.

The down-type quarks $d - s - b$ can be associated with $L = 1$:

$$T_{d-s} \approx T_{s-b} \approx 2\pi, \quad (18)$$

the up-type quarks $u - c - t$ with $L = 2$:

$$T_{u-c} \approx T_{c-t} \approx 4\pi. \quad (19)$$

The mass ratios between the different generations can therefore be reduced to a number of natural size.⁵ Although the approximation (14) is very rough and depends on the special parametrization, the factor 2 between the two different periods 2π and 4π is purely topological and does not depend on the special ansatz and parametrization.

⁵The leptons seem to have a mixing between both periods, which could happen by a variation of the corresponding coupling.

Why are there only three generations? We want to give some speculations why there are only a finite number of generations: The periodic solution cannot be the whole story for a theory with non-vanishing beta function. For a negative beta function⁶ asymptotic freedom has to be incorporated. At small distances the coupling gets small and perturbation theory should be valid. No dynamical mass generation is expected and the periodic solution should converge into the one loop expression. For large distances below the quantum scale Λ of the theory, where the coupling gets large, or below a possible current mass of the fermion the theory is no longer approximately scale invariant. But scale invariance was essential for arriving at the translation invariant differential equation (12) and the periodic solution. Periodicity of the solution is therefore only expected in a region of intermediate coupling. This region has finite size (for $\beta \neq 0$) and thus the number of generations is expected to be finite. Within this approach the value 3 for the number of generations is just a dynamical accident.

A crucial point, which we have not yet addressed, is the question for the underlying interaction responsible for that dynamical symmetry breaking scenario. To explain it by merely Standard Model interactions seems to be impossible. On the other side a new interaction is hard to introduce since the new bosons should be nearly massless. The gauge group must be hidden in some way to forbid vector boson interactions at tree level and to admit only self energy and vertex corrections. For the future an explicit model should be constructed.

4 Multipol propagator versus dynamical mass function

How can one explore whether some particles with the same quantum numbers really come from different propagators or belong to one propagator with several poles? One would naively suppose that there must be some essential differences. Starting with an exponential fermionic mass spectrum and cutting the spectrum as described in section 3, a theory of N_f free fermions remains. The sum of their propagators can be interpreted as one propagator with a mass function $B(p^2)$:

$$\frac{i}{\not{p} - B(p^2)} = \sum_{k=1}^{N_f} \frac{i}{\not{p} - m_k}, \quad \text{where } m_k := m_1 e^{\alpha(k-1)} \quad (20)$$

⁶For $\beta > 0$ the discussion is the same except for interchanging large and small distances.

As explained in the introduction for $N_f > 1$, $B(p^2)$ is no longer a well behaved selfenergy, but has poles. To avoid them each fermion self energy has to acquire an imaginary part as it appears e.g. in an one-loop calculation, if we admit an interaction:

$$\frac{i}{\not{p} - m_k} \rightarrow \frac{i}{\not{p} - m_k} + \frac{i}{\not{p} - m_k} (-i\Sigma_k) \frac{i}{\not{p} - m_k} \quad (21)$$

where

$$\Sigma_k = c \cdot i \cdot \theta(p^2 - m_k^2) \frac{p^2 - m_k^2}{p^2} \cdot m_k \quad (22)$$

with some constant c . The additional terms can be interpreted as some kind of continuum produced by that interaction. Extracting the \not{p} -independent part one can calculate the function $u(t)$ from $B(p^2) = e^{t/2}(u(t) + 1)$. It is plotted in fig.7. One obviously gets

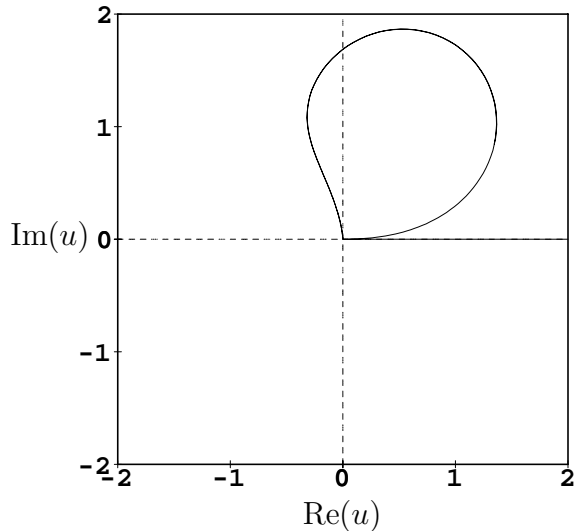


Figure 7 u from the sum of several independent propagators.

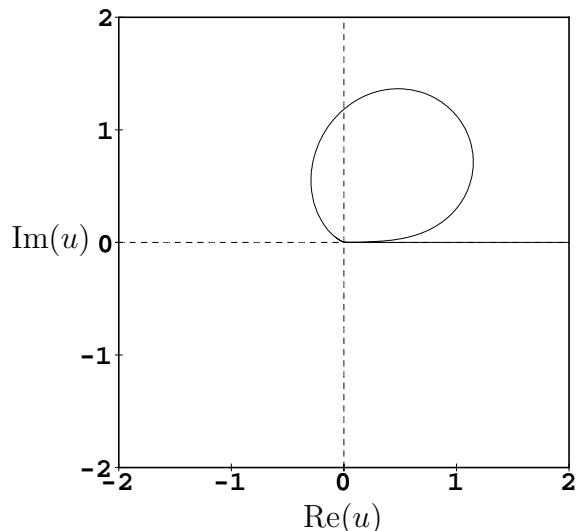


Figure 8 Solution of the gap equation with $L = 1$.

a periodic structure similar to fig.8. Our differential equation (13) with $\mathcal{C}(0) = 1.5$, $\alpha = -1.5$, $\beta = -0.453$ and $\gamma = 1.0$ yields that solution, where $L = 1$. One should not expect exact coincidence because we are comparing an one loop calculation with a solution of a much more complicated gap equation.

Hence our solution can effectively be described by the sum of independent propagators. A gauge field coupled to *the* fermion field seems to allow the decay of our states by going to different on-shell limits for an incoming and the corresponding outgoing fermion line. This is *not* true and we will prove it in the next section.

5 Effective Lagrangians and the stability of the mass spectrum

Starting from a scale invariant theory of one quark flavor interacting with some boson field we saw how to obtain multiple flavors with an exponential mass hierarchy. This has been achieved by analyzing gap equations for the fermion propagator S . As shown in the last section the solution takes the form

$$S(p) = \sum_{k=1}^{N_f} \frac{i}{\not{p} - m_k} + \text{continuum} \quad (23)$$

We ignore the continuum for a while, which describes some residual interaction. The effective Lagrangian

$$\mathcal{L}_{1,free}^{eff} = i\bar{\psi}S_0^{-1}\psi \quad , \quad S_0 = \sum_{k=1}^{N_f} \frac{i}{i\not{\partial} - m_k} \quad (24)$$

reproduces S (without continuum) and hence has the same particle spectrum. The Lagrangian is non-local due to the non-local kernel S_0^{-1} , but despite this it describes a free theory of pointlike particles, because it is quadratic in the fermion fields. The more familiar local Lagrangian

$$\mathcal{L}_{N_f,free} = \sum_{k=1}^{N_f} \bar{\psi}_k (i\not{\partial} - m_k) \psi_k \quad (25)$$

containing explicitly N_f fermion field operators ψ_k , one for each single particle state, describes also the same physics. One might interpret (25) as an effective Lagrangian of (24) which is itself (the free part of) an effective Lagrangian of the original theory.

There are two types of interactions which should be incorporated into eq.(24) now. One is some residual interaction of the mass-generating boson studied in the gap equation and coded in the continuum contribution to eq.(23). The other are interactions with other (standard model) gauge bosons. E.g. electromagnetism, when present in the original theory, must also emerge in some way in the effective theory. The principles of gauge invariance and minimal substitution ($\not{\partial} \rightarrow \not{D}$) dictates the form of the interaction in both cases:

$$\mathcal{L}_1^{eff} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + i\bar{\psi}S^{-1}\psi \quad , \quad S = \sum_{k=1}^{N_f} \frac{i}{i\not{D} - m_k} \quad , \quad \not{D} = \not{\partial} + ig\not{A} \quad (26)$$

\not{D} is the covariant derivative depending on the gauge field A_μ and $F_{\mu\nu}$ is its field strength. The scattering of n quarks can be obtained as usual from the $2n$ -point function. Integrat-

ing out *the*⁷ fermion field in the background of the gauge field A_μ we get the path integral representation for the $2n$ -point function⁸

$$\begin{aligned} & \langle 0 | \psi(x_1) \bar{\psi}(y_1) \dots \psi(x_n) \bar{\psi}(y_n) | 0 \rangle = \\ & = \int \mathcal{D}A_\mu e^{-i \int d^4x \frac{1}{4} F^2(x)} \text{Det} S^{-1} \prod_{l=1}^n \langle x_l | \sum_k \frac{i}{i \not{D} - m_k} | y_l \rangle + \text{crossed terms} \end{aligned} \quad (27)$$

For each contraction of a ψ with a $\bar{\psi}$, we get a propagator $\langle x | S | y \rangle$ in coordinate representation in the background of the gauge field A_μ . $\text{Det} S^{-1}$ is the functional determinant of the kernel. To get e.g. the propagator (23) one has to use eq.(27) with $n = 1$, which is just the integration of the propagator (26) over the quantum fluctuations of the gauge field.

It is instructive to compare eq.(27) to the gauged version of eq.(25), the standard model of N_f fermions coupled to a gauge field

$$\mathcal{L}_{N_f} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \sum_{k=1}^{N_f} \bar{\psi}_k (i \not{D} - m_k) \psi_k \quad (28)$$

with

$$\begin{aligned} & \langle 0 | \psi_{k_1}(x_1) \bar{\psi}_{k_1}(y_1) \dots \psi_{k_n}(x_n) \bar{\psi}_{k_n}(y_n) | 0 \rangle = \\ & = \int \mathcal{D}A_\mu e^{-i \int d^4x \frac{1}{4} F^2(x)} \prod_{k=1}^{N_f} \text{Det}(i \not{D} - m_k) \prod_{l=1}^n \langle x_l | \frac{i}{i \not{D} - m_{k_l}} | y_l \rangle + \text{crossed terms} \end{aligned} \quad (29)$$

The only difference between eq.(27) and eq.(29) are the functional determinants. At tree level, or more general in quenched approximation, both expressions coincide⁹:

$$\langle 0 | \psi(x_1) \bar{\psi}(y_1) \dots \psi(x_n) \bar{\psi}(y_n) | 0 \rangle = \sum_{k_1 \dots k_n} \langle 0 | \psi_{k_1}(x_1) \bar{\psi}_{k_1}(y_1) \dots \psi_{k_n}(x_n) \bar{\psi}_{k_n}(y_n) | 0 \rangle \quad (30)$$

In summary we can say that our model of mass generation can be described by the effective action \mathcal{L}_1^{eff} which reduces to \mathcal{L}_{N_f} in quenched approximation. Dynamical quark loops incorporated by the functional determinant lead to effective non-local boson vertices as in the standard theory \mathcal{L}_{N_f} , but differ from them in magnitude.

One immediate consequence of the discussion above is the absence of flavor changing currents. For \mathcal{L}_{N_f} this is obvious and therefore it must be true for \mathcal{L}_1^{eff} at tree level due to the equivalence proven above. Further, the inclusion of dynamical quark loops has

⁷ Note that there is still only one fermion field.

⁸ To get familiar with the operator notation and the background techniques one should consult [13, 14].

⁹ Taking the on-shell limit selects the appropriate flavor from the sum on the right hand side.

no influence on the flavor structure of the external fermions. This is consistent with the reality of the masses m_n , which also indicates the stability of the particles (but in a much less convincing way).

Feynman rules for the Lagrangian \mathcal{L}_{eff}^1 are derived in the appendix. They differ from the standard feynman diagrams of \mathcal{L}_{N_f} only by one additional flavor changing graph

$$k \rightarrow \times \rightarrow k' = -S_0^{-1} \quad (31)$$

connecting flavor k with k' , which is shown to be allowed only in closed fermion loops. This provides a perturbative proof of the stability of the mass spectrum. In the next section we will show, that everything said above applies to electromagnetism, weak and strong interaction, where ψ as well as each ψ_k have to be interpreted as a single fermion, a weak doublet or a color triplet respectively.

6 More realistic models

In this section we want to scetch what is changed when going from the toy model with one flavor per generation to more realistic models with two flavors per generation. We consider an $U(1) \times SU(2)$ gauge theory, which is not the Standard Model but rather a strongly coupled hidden gauge theory, with abelian coupling e , vector coupling g_V and axial coupling g_A

$$\mathcal{L}_{int} = e\overline{\psi^m}A\psi^m + \overline{\psi^m}(g_V - g_A\gamma_5)W_a\tau_{mn}^a\psi^n \quad , \quad m, n = 1, 2 \quad , \quad a = 1, 2, 3 \quad (32)$$

where summation over isospin indices m , n and a is understood. ψ is now an isospin doublet combining an up-type quark with a down-type quark ($\begin{smallmatrix} u \\ d \end{smallmatrix}$) (or a lepton with its neutrino ($\begin{smallmatrix} \nu_l \\ l \end{smallmatrix}$)). The propagators of both fermions can be combined into an isodoublet matrix like for the Dirac-operator.

For an isospin symmetric ansatz $S^{mn} = i\delta^{mn}/(\not{p} - B)$, the discussion in the previous chapters achieves only minor modifications. The gap equation (4) can be derived, with Γ_a replaced by $\Gamma_\mu = -ie\gamma_\mu$ and $\Gamma_\mu^a = -i(g_V - g_A\gamma_5)\gamma_\mu\tau^a$. We obtain the equation (9) with

$$C = (\Gamma_\mu\Gamma^\mu + \Gamma_\mu^a\Gamma_a^\mu)/(4\pi)^2 = -\frac{e^2}{4\pi^2} - 3 \cdot \frac{g_V^2 - g_A^2}{4\pi^2}. \quad (33)$$

Replacing the coupling constant C by a running coupling $\mathcal{C}(u)$ we get periodic solutions and hence an exponential mass spectrum, where the up-type quarks are degenerate in

mass with the down-type quarks. There is no isospin breaking, of course, because we have chosen a symmetric ansatz. The discussion of section 5 remains valid, if we interpret ψ and each ψ_k as an isodoublet and use $\mathcal{D} = \not{\partial} + ie\not{A} + i(g_V - g_A\gamma_5)\tau^a W_a$. Especially there are no FCNC and the CKM matrix is identical to $\mathbb{1}$.

An isovector ansatz $S = i/(\not{p} - B_a\tau^a)$ is also a solution of eq. (27). If we choose the orientation of $B_a(p^2)$ constant in isospace, say $B_a(p^2) = (0, 0, B(p^2))$ we arrive again at eq. (5) with C replaced by¹⁰

$$\tilde{C} = (\Gamma_\mu\tau^b\Gamma^\mu + \Gamma_\mu^a\tau^b\Gamma_a^\mu)/[(4\pi)^2\tau^b] = -\frac{e^2}{4\pi^2} + \frac{g_V^2 - g_A^2}{4\pi^2} \quad (34)$$

Again for a running coupling we get an isospin degenerate exponential mass spectrum, but now the up- and down-type masses differ in an irrelevant sign ($m_u = -m_d$).

The most general (parity even) ansatz

$$S = \frac{i}{\not{p} - B_0 - B_a\tau^a} \quad (35)$$

leads to a system of coupled differential equations

$$\begin{aligned} \ddot{y}_0 + 2\dot{y}_0 + \frac{3}{4}y_0 + \frac{C}{2} \left(\frac{y_0 - 1}{(y_0 - 1)^2 - \bar{y}^2} + \frac{y_0 + 1}{(y_0 + 1)^2 - \bar{y}^2} \right) &= 0 \\ \ddot{y}_a + 2\dot{y}_a + \frac{3}{4}y_a - \frac{\tilde{C}}{2} \left(\frac{y_a}{(y_0 - 1)^2 - \bar{y}^2} + \frac{y_a}{(y_0 + 1)^2 - \bar{y}^2} \right) &= 0 \end{aligned} \quad (36)$$

with $\bar{y}^2 = y_a y^a$ and C and \tilde{C} are defined above. The physical masses occur at the poles $(y_0 \pm 1)^2 = \bar{y}^2$. One should now study general solutions with running coupling and select those with the lowest vacuum energy. These solutions might provide an interesting isospin breaking pattern with an interesting CKM matrix in addition to the exponential mass spectrum.

When studying more general gauge groups, e.g. a GUT theory combining all fermions of one generation into a multiplet, eq.(36) remains valid. Only the definition of C and \tilde{C} and the number of components of y_a changes.

7 Conclusion

In this paper we gave a new approach to understand the Standard Model fermion mass spectrum. Since a possible horizontal symmetry must be broken drastically, we have

¹⁰Both signs of C and \tilde{C} can be realized as anticipated in section 2

instead investigated an improved gap equation with periodic solutions. This equation is triggered by a new strongly coupled boson and yield a fermion propagator with a series of poles. The scale invariant structure of the system automatically leads to the desired exponential particle spectrum. All particles have the same residue and couple to all bosons with the same strength. The mass ratios between particles of successive generations turn out to be

$$\frac{m_{n+1}}{m_n} = e^{L\pi\mathcal{O}(1)} \quad (37)$$

with $L = 1$ or 2 and can therefore be attributed to numbers of natural size. We found a good agreement on logarithmic accuracy with the spectrum of the up and down-type quarks of the Standard Model.

The stability of the excited fermions, which is the most difficult problem, is solved in the effective Lagrangian formalism. Gauge invariance dictates the structure of the effective theory which inevitably forbids flavor changing neutral transitions.

Because of this success the model seems to be a good candidate for the mass generation and especially to explain the exponential structure of the spectrum. A correct inclusion of the anomalous dimension of the coupling should lead to a finite number of generations as discussed in section 3. The next step will be to specify the new interaction and to determine the detailed structure of the mass generation. Then the effective Lagrangian can be extracted more rigorously from that specific model.

For the future it would also be of great interest to study loop corrections, which admit virtual flavor changing currents and hence lead to deviations from standard model predictions. Finally a complete model would offer the possibility to calculate the CKM matrix and therefore to test the theory accurately.

Appendix

A Feynman rules of the effective model

To determine the Feynman rules of the effective model discussed in section 5 we start with the partition function

$$Z[\bar{\eta}, \eta, j] = \int \mathcal{D}A_\mu \mathcal{D}\psi \mathcal{D}\bar{\psi} e^{i \int d^4x (\mathcal{L} + \bar{\eta}\psi + \bar{\psi}\eta + j^\mu A_\mu)} \quad (A.1)$$

where

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + i\bar{\psi}S^{-1}\psi \quad , \quad S = \sum_{k=1}^{N_f} \frac{i}{i\mathcal{D} - m_k} \quad , \quad \mathcal{D} = \not{\partial} + ig\not{A}. \quad (\text{A.2})$$

S^{-1} can be expanded using $\Delta_k := \frac{1}{i\not{\partial} - m_k}$ and $S_0 := i \sum_{k=1}^{N_f} \Delta_k$:

$$S^{-1} = S_0^{-1} + VS_0^{-1} \quad (\text{A.3})$$

with

$$V = \left[1 + iS_0^{-1} \sum_{k=1}^{N_f} \left(g\Delta_k\not{A}\Delta_k + g^2\Delta_k\not{A}\Delta_k\not{A}\Delta_k + \dots \right) \right]^{-1} - 1. \quad (\text{A.4})$$

Integrating out the fermion field we get

$$Z[\bar{\eta}, \eta, j] \propto \int \mathcal{D}A_\mu e^{i \int d^4x \left(-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + j^\mu A_\mu \right)} Z[\bar{\eta}, \eta; A] \quad (\text{A.5})$$

with

$$Z[\bar{\eta}, \eta; A] = e^{-i \int d^4x \frac{\partial}{\partial \bar{\eta}} iVS_0^{-1} \frac{\partial}{\partial \eta}} e^{i \int d^4x \bar{\eta} iS_0 \eta}. \quad (\text{A.6})$$

Terms, which contain one open fermion line and an arbitrary number of boson lines, can be extracted by an expansion in V . The n boson–fermion vertices can be ordered in $n!$ different ways. Thus the factors in the exponential series are compensated by combinatorical factors. We therefore get a geometric series:

$$\begin{aligned} Z_{open}[\bar{\eta}, \eta; A] &= -\bar{\eta}S_0 \left(1 - V + V^2 - V^3 + \dots \right) \eta \\ &= -i \sum_{k=1}^{N_f} \bar{\eta} \left(\Delta_k + g\Delta_k\not{A}\Delta_k + g^2\Delta_k\not{A}\Delta_k\not{A}\Delta_k + \dots \right) \eta. \end{aligned} \quad (\text{A.7})$$

There are no flavor changing transitions because there occurs Δ_k for only one flavor k in each term. This is due to the cancellation of the geometric series against the inversion in eq.(A.4). The series (A.7) can be summed up to $-i\bar{\eta}S\eta$, which is the tree level part of eq.(27).

Terms, which contain one closed fermion line and an arbitrary number of boson lines, are

$$\begin{aligned} Z_{closed}[\bar{\eta}, \eta; A] &= -i\text{Tr} \left[\left(V - \frac{1}{2}V^2 + \frac{1}{3}V^3 - \dots \right) \right] \\ &= ig^2\text{Tr} \left[\sum_{k=1}^{N_f} \Delta_k\not{A}\Delta_k\not{A}\Delta_k iS_0^{-1} - \frac{1}{2} \sum_{k,j=1}^{N_f} \Delta_k\not{A}\Delta_k iS_0^{-1} \Delta_j\not{A}\Delta_j iS_0^{-1} \right] + \mathcal{O}(g^4). \end{aligned} \quad (\text{A.8})$$

For a closed fermion loop n boson–fermion vertices can be ordered in $(n-1)!$ different ways. Hence we find a logarithmic series, which gives $-i\text{Tr} \ln(1+V) = -i \ln \text{Det}(1+V)$.

The sum over all diagrams with an arbitrary number of loops is proportional to $\text{Det}S^{-1}$, which is the loop correction in eq.(27). The mixed occurrence of $\Delta_k, \Delta_j, \dots$ in eq.(A.8) shows that flavor changing transitions appear in closed fermion lines. The transition is induced by the graph (31).

The Feynman rules can be read of eq.(A.7) and eq.(A.8). They only differ in the loop corrections from the standard model (28). The spectrum is therefore stable.

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